TECHNIQUES FOR COMPUTING EXACT HAUSDORFF MEASURE WITH APPLICATION TO A SIERPINSKI SPONGE IN $\mathbb{R}^3$

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University of St Andrews

30th March 2006

For Mum and Dad.

Declaration

I, Barry Martin Ridge, declare that this thesis has been composed by myself, that it is a record of my own work, and that it has not been accepted in any previous application for any degree.

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Abstract

In this dissertation we aim to perform a detailed study of techniques for the analysis of the exact s-dimensional Hausdorff measure of fractal sets and try to provide a reasonably comprehensive review of the required background. An emphasis is placed on results pertaining to local density of sets and we show how these provide a link to the more global concept of Hausdorff measure. A new result is provided which states that if K is a self-similar set satisfying the open set condition, then $\mathcal{H}^s(K \cap U) \leq |U|^s$ for all Borel U, also implying that $\overline{D}_c^s(K,x) \leq 1$ for all x, where $\mathcal{H}^s(E)$ and $\overline{D}_c^s(E,x)$ refer to the s-dimensional Hausdorff measure of some set E and the local convex density of E at a point x respectively. Based on the work of Zuoling Zhou and Min Wu, we provide new calculations for the exact Hausdorff measure of both a Sierpinski carpet in $\mathbb{R}^2$ and a Sierpinski sponge in $\mathbb{R}^3$. In the final chapter we take a look at how the Hausdorff measure behaves when measuring the invariant sets associated with special types of iterated function systems known as iterated function systems with condensation and also provide a brief discussion on the calculation of the packing measure of a self-similar set.

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Introduction

0.1 Notational Conventions

Some points regarding notational conventions used in the sequel:

We will refer to a ball with its centre at a point x and a radius r as either $B_r(x)$ or $B(x, r)$ interchangeably.

Also, $\overline{\lim}$ and $\underline{\lim}$ will refer to upper and lower limits respectively.

0.2 Summary

Calculating the Hausdorff measure of fractal sets is an inherently difficult problem owing to the very definition of Hausdorff measure: the sum of the diameters of all the sets of various sizes and shapes that make up the most efficient cover of a given set to be measured. The fact that covering sets are allowed to vary so much means that there is an extremely wide class of covers to consider when finding the most efficient one. This stands in contrast to the class of covers used when calculating box-counting dimension, where only covering sets of a fixed size and shape are considered.

The first two chapters of this dissertation provide a historical discussion of the necessary concepts from measure theory and fractal geometry, and a review of iterated function systems and self-similar sets respectively. The first chapter begins with a brief discussion of the first basic efforts to define a measure that assigns a 'length' value to arbitrary sets in $\mathbb{R}$, and attempts to track the progression to the more robust notion of Hausdorff measure. This path of development traverses a number of important junctions, including sigma-algebras, metric outer measures and Methods I and II for the construction of metric outer measures. Toward the end of the chapter, we arrive at a definition of Hausdorff measure $\mathcal{H}^s(E)$ for a given set $E \in \mathbb{R}^d$ and discuss how its namesake, Felix Hausdorff, discovered that for any value of s other than a certain critical value that pertains to the set E being measured, $\mathcal{H}^s(E)$ will always be either 0 or $+\infty$. This critical value for s is the Hausdorff dimension of the set E. It is capable of taking on non-integral values and is frequently used to gauge the 'complexity' of the set. We discuss how calculating the Hausdorff dimension directly can be tricky, but is made easier through the use of the more accessible box-counting dimension and its calculation.

In the second chapter we talk about iterated function systems or IFSs and self-similar sets. Iterated function systems are quite important as they facilitate the definition of a broad class of fractal sets; indeed, most of the fractal sets that may be found in today's books and papers on fractals are generated using iterated function systems. For example, the classic middle-third Cantor set may be generated using the IFS

$$\left\{f_1(x) = \tfrac{1}{3}x, f_2(x) = \tfrac{1}{3}x + \tfrac{2}{3}\right\}.$$

IFSs are constructed using contraction mappings with associated contraction ratios or Lipschitz constants. In the above example, both f_1 and f_2 are contraction mappings with contraction ratios $\frac{1}{3}$. An IFS always has a unique invariant set associated with it, often referred to as the attractor or fixed point of the IFS, which is generated by iterating the collection of mappings contained in the IFS over any given set infinitely many times. Given an IFS $\{S_1, S_2, S_3, S_4\}$, the invariant set associated with that IFS is given by

$$F = \bigcup_{i=1}^{4} S_i(E)$$

where E is a given set in the space that we are working in. We pay particular attention to a certain type of IFS in Chapter 2, namely iterated function systems with condensation. These are regular IFSs, but some fixed set called a ‘condensation set’ is merged with the output of the IFS at each iteration when constructing the invariant set. There is a classical result associated with IFSs which shows that all IFSs have a unique invariant set associated with them. In Chapter 2, we prove this result for IFSs with condensation; the proof is not much different to the proof for the regular case, but is not seen as often in the literature so we decided to prove this version here.

As was mentioned in the abstract, a number of results can be garnered for the local density of a set at a given point which can form the basis for the calculation of the exact Hausdorff measure of the set at the critical dimension. Several attempts at calculating the Hausdorff measure in such a way have been made by various authors for various different fractal sets. We provide a brief review of some of these attempts at the end of Chapter 3, after analysing many of the key results for local density. Two types of local density that are of particular interest to us are local spherical density and local convex density. The upper spherical density with respect to the Hausdorff measure of a set E, with positive finite Hausdorff dimension s, at a point x is

$$\overline{D}^s(E, x) = \overline{\lim_{r \to 0}} \frac{\mathcal{H}^s(E \cap B_r(x))}{(2r)^s}.$$

The upper convex density for E at x is

$$\overline{D}_c^s(E, x) = \overline{\lim_{r \to 0}} \left\{ \sup \frac{\mathcal{H}^s(E \cap U)}{|U|^s} \right\},$$

where the supremum is over all convex sets U with $x \in U$ and $0 < |U| < r$. As we point out in the chapter, local spherical density is not quite as useful as local convex density with respect to the Hausdorff measure. In particular, one of the main results obtainable for spherical density states that $2^{-s} \leq \overline{D}^s(E,x) \leq 1$ for $\mathcal{H}^s$-almost all x, but the equivalent result for convex density states that $\overline{D}_c^s(E,x) = 1$ for $\mathcal{H}^s$-almost all x which is a much more useful result in practice. After taking a look at the proofs of these two results, we provide a new result for local convex density in Section 3.3.3, namely that given a self-similar set K satisfying the open set condition

$$\mathcal{H}^s(K \cap U) \leq |U|^s$$

for all Borel U, thus $\overline{D}_c^s(K,x) \leq 1$ for all x. This proves quite useful for attaining upper bounds in the proofs of some further results later in the chapter. These culminate in Theorem 3.3.13, which states that given a self-similar set K satisfying the open set condition and a suitable self-similar measure λ supported on K, then

$$\mathcal{H}^s(K) = \frac{1}{\sup_x \overline{d}_c^s(\lambda, x)}$$

where $\overline{d}_c^s(\lambda, x)$ refers to the upper convex density at a point x with respect to λ.

In Chapter 4, based on the work of Zhou and Wu in [ZW99], we present a calculation for the exact s-dimensional Hausdorff measure of a Sierpinski carpet in $\mathbb{R}^2$. The particular Sierpinski carpet analysed is the invariant set associated with an iterated function system consisting of four similarity mappings, each of which rescales sets by a factor of $\frac{1}{4}$. When acting on the unit square in $\mathbb{R}^2$, each of the mappings maps to one of the corners of the unit square. We make a number of modifications to the method used by Zhou *et al* which simplify the calculation considerably. Referring to the Sierpinski carpet in question as C, we prove that $\dim_{\mathcal{H}} C = 1$ and that $\mathcal{H}^1(C) = \sqrt{2}$. The upper bound for the Hausdorff measure calculation is found by using a theorem due to Hutchinson [Hut81] which states that if C is a self-similar set generated by an IFS with mappings $\{R_1, \ldots, R_n\}$ with associated contraction ratios $\{c_1, \ldots, c_n\}$, letting s be a unique real number such that $\sum_{i=1}^{n} c_i^s = 1$, then we have $\mathcal{H}^s(C) \leq \text{diam}(K)^s$. We use the mass distribution principle to ascertain the lower bound. Using an appropriate mass distribution μ supported on the set

C, if we can show that $\mu(V) \leq |V|$ for all measurable sets V, then $\mathcal{H}^s(C) \geq \mu(C)$ and we are done. Central to the proof is the idea of projecting the set C onto one of its main diagonals and defining a mass distribution m supported on the projection which is based on the original mass distribution supported on C. Zhou *et al* use a number of lemmas to show that $m([0,x]) \geq \frac{1}{2}x$ for all $x \in [0, \sqrt{2}]$. We condense this into a single lemma and improve upon the result slightly, showing that $m([0,x]) \geq \frac{4}{7}x$ for all $x \in [0, \sqrt{2}]$. Developing the proof for this result was aided by the graph of $y = m([0,x])$ and $y = \frac{4}{7}x$ shown in Figure 4.5.1.

We extend the Hausdorff measure calculation from Chapter 4 to a three-dimensional case in Chapter 5, analysing a Sierpinski Sponge in $\mathbb{R}^3$ which may be generated by an IFS consisting of 8 contraction mappings of Lipschitz ratio $\frac{1}{8}$ which map the unit cube to $\frac{1}{8}$-scaled copies of itself in each of its 8 corners. Letting C denote the Sierpinski carpet in $\mathbb{R}^3$ this time, we prove that $\dim_{\mathcal{H}} C = 1$ and that $\mathcal{H}^1(C) = \sqrt{3}$. The method used is largely the same as that of Chapter 4 and the calculations are not complicated too much further by the addition of a third dimension to the space we are working in.

In the final chapter we take a look at how the Hausdorff measure behaves when measuring the invariant sets associated with iterated function systems with condensation. We make an interesting observation which shows that the Hausdorff measure changes from being a positive finite value to being $+\infty$ when measuring the invariant sets associated with two different IFSs with condensation which differ only very slightly. We also take a look at the packing measure, a notion of measure which has risen in status next to the Hausdorff measure in recent years and is now regarded as being equally important. Packing measure is defined in a similar way to the Hausdorff measure, but uses efficient packings of sets as opposed to efficient covers in its definition. In the final section of the chapter, we discuss the work done by Jia *et al* in [JZZL03] on the calculation of the packing measure of the Cartesian product of the middle third Cantor set with itself in $\mathbb{R}^2$ and show how they make use of some results that link local spherical density to the packing measure in order to achieve their result.

Chapter 1

Measure and Dimension

1.1 Introduction

The journey to a notion of sets of infinite complexity, with non-integral dimensions and self-similar properties began in a setting that, to an non-mathematician, might seem slightly strange. Toward the end of the 19th century, the world of pure mathematics had encountered a problem. Riemann integration, although quite successful in dealing with many functions such as continuous functions and functions on closed bounded intervals, failed to deal with more irregular functions such as limiting processes. The French mathematician Henri Lebesgue saw that there was work to be done in this area and in 1901, he formulated a theory of measure which extended Riemann's theory of integration to allow for the possibility of more irregular functions. Lebesgue's nuance was centred on the concept of length. How does one define the 'length' of an arbitrary set in $\mathbb{R}$? The following definition illustrates what we might think of as an intuitive description of an idealised length function:

Definition 1.1.1. A function $\ell : \{A | A \subseteq \mathbb{R}\} \to [0, \infty]$ is called a *length function* if

1. $\ell(\emptyset) = 0$
2. $A \subseteq B \Rightarrow \ell(A) \leq \ell(B)$
3. If $A_1, A_2, \ldots \subseteq \mathbb{R}$ are sets such that $A_i \cap A_j = \emptyset$ for $i \neq j$ (pairwise disjoint), then $\ell(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \ell(A_k)$ (countable additive).
4. If $A \subseteq B$ and $x \in \mathbb{R}$, then $\ell(A) = \ell(A + x)$ (translation invariance).
5. $\ell([0, 1]) = 1$

Unfortunately, as was shown by Vitali, such a length function does not exist. A possible solution arises by replacing the 3rd countable additivity condition with a finite additivity condition: $\ell(\bigcup_{k=1}^{n} A_k) = \sum_{k=1}^{n} \ell(A_k)$. However, a counter-example known as the Banach-Tarski paradox was found in the 1920's which disproves such a modified length function for dimensions ≥ 2. Lebesgue, instead, introduced a concept known as *measure* which involved countable subadditivy instead of countable additivity. This allowed him to formulate a general concept of 'length' for 1-dimensional sets which facilitated the introduction of Lebesgue integration. Subsequently Constantin Carathéodory, a German mathematician of Greek descent, became interested in extending measure theory to n-dimensional cases. His efforts were successful and formed the basis for the discovery of another German mathematician, Felix Hausdorff, of the existence of non-integral dimensions. In order to track how Carathéodory did this we will need a number of definitions.

Lebesgue's original concept of measure involved notions of *outer measure* and *inner measure*. A set was said to be *Lebesgue-measurable* if its outer measure and inner measure coincided. Carathéodory's measure theory dispensed with inner measure and provided an alternative, non-intuitive definition of measurability which proved to be the key underpinning of Hausdorff's subsequent work.

1.1.1 Outer Measure

Definition 1.1.2. A function $\mu : \{A | A \subseteq \mathbb{R}^d\} \to [0, \infty]$ is called an *outer measure* if

1. $\mu(\emptyset) = 0$
2. $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$
3. If $A_1, A_2, \ldots \subseteq \mathbb{R}^d$, then $\mu(\bigcup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} \mu(A_k)$ (countable subadditive).

Carathéodory devised the following δ-approximative outer measure to deal with n-dimensional sets $E \subseteq \mathbb{R}^d$:

$$\mathcal{C}_\delta(E) = \inf \left\{ \sum_{i=1}^{\infty} \text{diam}(E_i) \,\middle|\, E \subseteq \bigcup_{i=1}^{\infty} E_i \,,\ \text{diam}(E_i) < \delta \right\} \tag{1.1.1}$$

As δ decreases, the number of ways in which we can cover E with suitable E_i sets is reduced. As that class of potential covers gets smaller, the infimum (smallest sum of covering sets) either remains the same or gets bigger as the options for efficient covers run out. So, as δ approaches zero, the infimum approaches a limit, leading us to the following definition:

$$\begin{aligned} \mathcal{C}(E) &= \lim_{\delta \to 0} \mathcal{C}_\delta(E) \\ &= \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} \text{diam}(E_i) \,\middle|\, E \subseteq \bigcup_{i=1}^{\infty} E_i \,,\ \text{diam}(E_i) < \delta \right\} \\ &= \sup_{\delta > 0} \mathcal{C}_\delta(E) \end{aligned}$$

Using his novel definition of measurability, Carathéodory went on to show that his outer measure actually fulfills the criteria for a ‘length function’ when applied to a certain class of sets known as *Borel sets*. Defining this class of sets requires the following definitions and results:

1.1.2 σ-Algebras

Definition 1.1.3. A family $\mathcal{A}$ of subsets of a set $X \subseteq \mathbb{R}^d$ is called an *algebra* if:

(i) $X \in \mathcal{A}$,

(ii) $A \in \mathcal{A} \Rightarrow A' \in \mathcal{A}$,

(iii) $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$,

where A' is the complement of A.

Lemma 1.1.4. *If $\mathcal{A}$ is an algebra of subsets of some set $X \subseteq \mathbb{R}^d$, then*

(1) $\emptyset \in \mathcal{A}$,

(2) $A_1, \dots, A_n \in \mathcal{A} \Rightarrow \bigcup_{i=1}^n A_i \in \mathcal{A}$,

(3) $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$,

(4) $A_1, \dots, A_n \in \mathcal{A} \Rightarrow \bigcap_{i=1}^n A_i \in \mathcal{A}$,

(5) $A, B \in \mathcal{A} \Rightarrow A \setminus B \in \mathcal{A}$.

Proof. (1) follows from (i) and (ii) of Definition 1.1.3. (2) follows by repeated application of (iii). Since $A \cap B = (A' \cup B')'$, (3) follows from (ii) and (iii). (4) comes from repeated application of (3). For (5), note that $A \setminus B = A \cap B' \in \mathcal{A}$ by (ii) and (3). □

Definition 1.1.5. An algebra $\mathcal{A}$ of subsets of a set $X \subseteq \mathbb{R}^d$ is called a *σ-algebra* if, in addition to the conditions for an algebra in Definition 1.1.3, the following condition is also satisified:

$$A_1, A_2, \dots \in \mathcal{A} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}.$$

The σ-algebra of an algebra $\mathcal{A}$ may be denoted $\sigma(A)$.

Definition 1.1.6. We will call a σ-algebra $\mathcal{A}$ in $\mathbb{R}^d$ *"good"* if $\mathcal{A}$ contains all open rectangles $(a_1, b_1) \times \ldots \times (a_n, b_n)$. E.g. $\mathcal{A} = \{A | A \subseteq \mathbb{R}^d\}$ is good.

Note: ”Good” is not a standard term for this definition, but will serve our purposes here.

Using the above definitions, we my now define the Borel sets as follows:

Definition 1.1.7. The intersection $\mathcal{B} = \bigcap_{n=1}^{\infty} \mathcal{A}$ of all good sets $\mathcal{A}$ is called the *Borel σ-algebra*.

The Borel σ-algebra describes an extremely wide class of sets. Any set that can be constructed using a sequence of countable unions or intersections starting with the open sets or closed sets will be Borel. This is more than adequate for our purposes, as the fractal sets we will be working with may be described in such a way.

Now that we know what the Borel sets look like, we can proceed and show that $\mathcal{C}$, acting on those sets, behaves like a ‘length’ function. Before doing that, we shall refine our notion of what a ‘length’ function should be. The definition of a *measure* presented in the next subsection is quite similar to the definition of a ‘length’ function, but uses a σ-algebra as its domain. We would like to show that $\mathcal{C}$ satisfies the criteria for this modified notion of measure when acting on the Borel sets.

1.1.3 Metric Outer Measure

The definition of measure that follows helps us reclaim the valuable countable-additive property which was sacrificed for countable-subadditivity in our definition of outer measure. This new type of measure usually operates on a slightly smaller class of sets, namely the Borel sets, as opposed to the entire family of subsets of $\mathbb{R}^d$ for outer measure.

Definition 1.1.8. Let $\mathcal{A}$ be a σ-algebra. A function $\mu : \mathcal{A} \rightarrow [0, \infty]$ is a *measure* if:

1. $\mu(\emptyset) = 0$
2. $A, B \in \mathcal{A}$, $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$
3. $A_1, A_2, \ldots \in \mathcal{A}$, $A_n \cap A_m = \emptyset$ for $n \neq m$ (pairwise disjoint), then $\mu(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \mu(A_k)$ (countable additive).

We follow this with a key theorem for measurability first introduced as a concept in Carathéodory's seminal 1914 paper [Car14] entitled *"Über das lineare Maß von Punktmengen- eine Verallgemeinerung des Längenbegriffs"* or *"On the Linear Measure of Point Sets- a Generalization of the Concept of Length"*. This theorem asserts the existence of a certain σ-algebra associated with any outer measure μ and says that μ is a measure on that σ-algebra. As was noted by Hewitt and Stromberg in [HS65], exactly how Carathéodory came up with this is quite mysterious as it is not at all intuitive. The important thing is that it works.

Theorem 1.1.9. (Carathéodory Extension Theorem) *Let μ be an outer measure. Put $\mathcal{A}(\mu) = \mathcal{A} = \{A \mid \forall E : \mu(E) = \mu(A \cap E) + \mu(E \setminus A)\}$. Then,*

1. *$\mathcal{A}$ is a σ-algebra.*
2. *$\mu : \mathcal{A} \rightarrow [0, \infty]$ is a measure.*

$\mathcal{A}$ is called the σ-algebra of μ-measurable sets.

Proof. A proof for this can be found in [Bar66] pages 101-103. □

Lemma 1.1.10. *Given an outer measure μ and some set $A \subseteq \mathbb{R}^n$, if $\mu(A) = 0$, then $A \in \mathcal{A}(\mu)$.*

Proof. Consider a set $E \subseteq \mathbb{R}^n$. By the second property of outer measure (monotonicity),

$$\mu(A \cap E) + \mu(E \setminus A) \leq \mu(A) + \mu(E) = \mu(E).$$

The third property of outer measure (subadditivity) yields the opposite inequality

$$\mu(A \cap E) + \mu(E \setminus A) \geq \mu(E),$$

so

$$\mu(A \cap E) + \mu(E \setminus A) = \mu(E)$$

and $A \in \mathcal{A}(\mu)$ by the Carathéodory extension theorem.

□

Carathéodory's Extension Theorem was the first step in showing that $\mathcal{C}$ is a measure on the Borel sets. The second step requires the notion of a *metric outer measure*. As the definition below and the theorem that follows it show, if an outer measure μ is a metric outer measure, then the Borel sets form a subset of its associated σ-algebra.

Definition 1.1.11. When $A, B \subseteq \mathbb{R}^d$ and dist$(A, B) = \inf_{a \in A, b \in B} |a - b|$, μ is called a *metric outer measure* if:

$$\mu(A \cup B) = \mu(A) + \mu(B) \;\; \forall\, A, B \text{ such that } \mathrm{dist}(A, B) > 0$$

Theorem 1.1.12. *If μ is a metric outer measure, then $\mathcal{B} \subseteq \mathcal{A}(\mu)$.*

Proof. Omitted. A proof for this may be found in [Fal86] on Page 6. □

We require the following small lemma later on in Chapter 3.

Lemma 1.1.13. *Let μ be a measure on a σ-algebra $\mathcal{E}$. Given sets $A_1, \ldots, A_n \in \mathcal{E}$, if $\mu(A_i \cap A_j) = 0$ when $i \neq j$, then*

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i).$$

Proof. This proof is omitted as it is a well known result and is relatively straightforward using basic set theory and the properties of measure. □

Using the definitions and theorems that preceded, we can prove that Carathéodory's outer measure $\mathcal{C}$ is a a metric outer measure and thus a measure on the Borel sets, or more succinctly, a *Borel Measure*.

Theorem 1.1.14. *$\mathcal{C}$ is a metric outer measure.*

Proof. Choose A, B such that dist$(A, B) = \delta > 0$. We want to show that $\mathcal{C}_\delta(A \cup B) = \mathcal{C}_\delta(A) + \mathcal{C}_\delta(B)$.

"$\leq$" We have $\mathcal{C}_\delta \leq \mathcal{C}_\delta(A) + \mathcal{C}_\delta(B)$ from Property 3 of outer measures in Definition 1.1.2.

"$\geq$" Let $\delta > 0$ such that $\delta <$ dist(A, B).

Let $\mathcal{D} = \bigcup_{i=1}^{\infty} D_i$ be any countable cover of $A \cup B$ such that diam$(D_i) < \delta$. diam$(D_i) <$ dist(A, B) for all i, thus each D_i set intersects at most one of either A or B, so we can split $\mathcal{D}$ into two disjoint collections, $\mathcal{D}_1$ and $\mathcal{D}_2$ covering A and B respectively. Thus,

$$\sum_{D_i \in \mathcal{D}} \text{diam}(D_i) = \sum_{D_i \in \mathcal{D}_1} \text{diam}(D_i) + \sum_{D_i \in \mathcal{D}_2} \text{diam}(D_i) \geq \mathcal{C}_\delta(A) + \mathcal{C}_\delta(B).$$

Taking infimum over all covers, we have

$$\inf \left\{ \sum_{D_i \in \mathcal{D}} \text{diam}(D_i) \;\middle|\; A \cup B \subseteq \mathcal{D} \right\} \;=\; \mathcal{C}_\delta(A \cup B) \;\geq\; \mathcal{C}_\delta(A) + \mathcal{C}_\delta(B)$$

and taking the limit as $\delta \to 0$ we get $\mathcal{C}(A \cup B) \;\geq\; \mathcal{C}(A) + \mathcal{C}(B)$.

□

Later in this dissertation we will require the following theorem for uniqueness of a measure on a σ-algebra.

Theorem 1.1.15. (Carathéodory Uniqueness Theorem or Hahn Extension Theorem) *Let $X \subseteq \mathbb{R}^d$ and let $\mathcal{A}$ be an algebra of subsets of X. Let μ and υ be finite measures on $\sigma(\mathcal{A})$ and let*

$$\mu(A) = \upsilon(A) \;\; \textit{for all} \;\; A \in \mathcal{A}.$$

Then

$$\mu(B) = \upsilon(B) \;\; \textit{for all} \;\; B \in \sigma(\mathcal{A}).$$

Proof. A proof for this may be found in [Bar66] pages 103-104. □

1.2 Construction of Outer Measures and Metric Outer Measures

There is a way of generalising the construction of outer measures known as Method I. Method I can be extended to generalise the construction of metric outer measures. This extension is known as Method II. Both of these methods will be useful to us when defining measures such as the Hausdorff measure which shall be discussed in the sequel.

1.2.1 Method I Outer Measures

Definition 1.2.1. Let $\mathcal{M}$ be a family of subsets such that $\mathbb{R}^d = \bigcup_{M \in \mathcal{M}} M$. Let $\mathcal{T} : \mathcal{M} \to [0, \infty]$ be any function. Define

$$\mu(A) = \inf \left\{ \sum_{i=1}^{\infty} \mathcal{T}(M_i) \,\middle|\, A \subseteq \bigcup_{i=1}^{\infty} M_i,\ M_i \in \mathcal{M} \right\}.$$

μ is called the *Method I Outer Measure* associated with $(\mathcal{M}, \mathcal{T})$.

Proposition 1.2.2. *μ is an outer measure.*

Proof.

1. $\mu(\emptyset) = 0$.

   This is obvious since the empty set is covered by the empty set and the empty sum is zero.

2. $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$.

   Fix $A \subseteq B$. Let $B \subseteq \bigcup_{i=1}^{\infty} M_i,\ M_i \in \mathcal{M}$. Then, $A \subseteq B \subseteq \bigcup_{i=1}^{\infty} M_i$ implies that:

$$\mu(A) \quad \leq \quad \sum_{i=1}^{\infty} \mathcal{T}(M_i) \ \text{ which is true for all such covers of } B\text{, hence}$$

$$\begin{aligned} \mu(A) &\leq \inf\left\{ \sum_{i=1}^{\infty} \mathcal{T}(M_i) \,\middle|\, B \subseteq \bigcup_{i=1}^{\infty} M_i,\ M_i \in \mathcal{M} \right\} \\ &= \mu(B) \end{aligned}$$

3. If $A_1, A_2, \ldots \subseteq \mathbb{R}^d$, then $\mu(\bigcup_{k=1}^{\infty}) \leq \sum_{k=1}^{\infty} \mu(A_k)$.

Case 1: $\mu(A_k) = \infty$ for one or more of the A_k.

If one of the A_k has measure infinity, then the sum of the measures of all the A_k will have measure infinity, which is always bigger than the left-hand side.

Case 2: $\mu(A_k) < \infty$ for all k.

Let $A_1, A_2, \ldots \subseteq \mathbb{R}^d$. Let $\epsilon > 0$ and fix $n \in \mathbb{N}$. It suffices to show that

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n) + \epsilon\,.$$

Since $\epsilon > 0$ and $\mu(A_n) < \infty,\ \mu(A_n) < \mu(A_n) + \frac{\epsilon}{2^n}$. There exists a cover $\bigcup_{i=1}^{\infty} M_{n,i}$ over A_n, where $M_{n,i} \subseteq \mathcal{M}$ such that

$$\sum_{i=1}^{\infty} \mathcal{T}(M_{n,i}) < \mu(A_n) + \frac{\epsilon}{2^n}$$

Since we can find such covers for all of the A_n, we have

$$\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty}\bigcup_{i=1}^{\infty} M_{n,i}.$$

The measure on the union of the A_n uses the most efficient cover, thus

$$\begin{aligned} \mu\left(\bigcup_{n=1}^{\infty} A_n\right) &\leq \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} \mathcal{T}(M_{n,i}) \\ &\leq \sum_{n=1}^{\infty} \left(\mu(A_n) + \frac{\epsilon}{2^n}\right) \end{aligned}$$

$$= \sum_{n=1}^{\infty} \mu(A_n) + \epsilon \sum_{n=1}^{\infty} \frac{1}{2^n}$$
$$= \sum_{n=1}^{\infty} \mu(A_n) + \epsilon$$

□

Unfortunately, as we shall now see, we can find a counter-example which shows that Method I outer measures are not always metric. The proof requires the Lebesgue outer measure, which we shall now define using Method I and a small, but useful theorem which shows that the Lebesgue outer measure on an interval is equal to the length of the interval.

Definition 1.2.3. Let $d = 1$. Let $\mathcal{M} = \{[a, b] \mid a < b\}$. Let $\mathcal{T}([a, b]) = b - a$. Then the *Lebesgue outer measure,* $\mathcal{L}$, is the Method I outer measure associated with $(\mathcal{M}, \mathcal{T})$.

Theorem 1.2.4. *If A is an interval, then $\mathcal{L}(A)$ is equal to the length of A.*

Proof. A proof for this may be found in [Yeh00], pages 36-37. □

Proposition 1.2.5. *Method I outer measures are not always metric.*

Proof. Let d = 1. Let $\mathcal{M} = \{[a, b) \mid a < b\}$. Let $\mathcal{T}([a, b]) = \sqrt{b - a}$. Let μ be the Method I outer measure associated with $(\mathcal{M}, \mathcal{T})$. Let $A = [-1, -\frac{1}{4})$ and let $B = [\frac{1}{4}, 1)$. First we will show that $\mu(A) = \frac{\sqrt{3}}{2}$:

"$\leq$" $\mu\left([-1, -\frac{1}{4})\right) \leq \mathcal{T}\left([-1, -\frac{1}{4})\right) = \sqrt{-\frac{1}{4} + 1} = \frac{\sqrt{3}}{2}$

"$\geq$" Let $[-1, -\frac{1}{4}) \subseteq \bigcup_{i=1}^{\infty} [a_i, b_i)$. Then

$$\left(\sum_{i=1}^{\infty} \sqrt{(b_i - a_i)}\right)^2 = \sum_{i,j=1}^{\infty} \sqrt{(b_i - a_i)}\sqrt{(b_j - a_j)}$$
$$= \sum_{i=1}^{\infty} \sqrt{(b_i - a_i)}\sqrt{(b_i - a_i)} +$$

$$
\begin{aligned}
& \sum_{i=1, i\neq j}^{\infty} \sqrt{(b_i - a_i)}\sqrt{(b_j - a_j)} \\
\geq \ & \sum_{i=1}^{\infty} \sqrt{(b_i - a_i)}\sqrt{(b_i - a_i)} \\
= \ & \sum_{i=1}^{\infty} (b_i - a_i) \\
\geq \ & \mathcal{L}(A) = |A| = \frac{3}{4}
\end{aligned}
$$

Therefore, we have $\sum_{i=1}^{\infty} \sqrt{(b_i - a_i)} \geq \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$.

It can be shown that $\mu(B) = \frac{\sqrt{3}}{2}$ in a similar way. Note that $\text{dist}(A, B) = \frac{1}{2}$, thus satisfying the preliminary requirement for the metric outer measure test. Summing the measures of A and B, we get $\mu(A) + \mu(B) = \sqrt{3}$. However, since $A \cup B \subseteq [-1, 1)$, we have

$$
\begin{aligned}
\mu(A \cup B) &= \mu(A) + \mu(B) \\
&\leq \mu([-1, 1)) \\
&\leq \mathcal{T}([-1, 1)) \\
&= \sqrt{1 - (-1)} = \sqrt{2}
\end{aligned}
$$

Hence $\mu(A \cup B) \neq \mu(A) + \mu(B)$, so the outer measure is not metric. □

1.2.2 Method II Outer Measures

We will now extend the notion of Method I outer measures to Method II outer measures, which can be shown to be metric and in particular, form measures on the Borel sets.

Definition 1.2.6. Let $\mathcal{M}$ be a family of subsets of $\mathbb{R}^d$ such that $\bigcup_{M\in\mathcal{M}} M = \mathbb{R}^d$ and let

$$\mathcal{M}_\delta = \{\, M \in \mathcal{M} \mid \text{ diam}\, M \leq \delta\} \, .$$

Let $\mathcal{T} : \mathcal{M} \to [0,\infty]$ be any function and let $\mathcal{T}_\delta = \mathcal{T}|_{\mathcal{M}_\delta}$. Let μ_δ be the Method I outer measure associated with $(\mathcal{M}_\delta, \mathcal{T}_\delta)$. Then the *Method II outer measure* is defined as follows:

$$\mu(A) = \lim_{\delta\to 0} \mu_\delta(A) = \sup_{\delta > 0} \mu_\delta(A).$$

Theorem 1.2.7. *Method II outer measures are metric outer measures.*

Proof. This is almost identical to the proof of Theorem 1.1.14 to show that Carathéodory's measure is a metric outer measure, so will be omitted, but it may be found in [Edg90] on page 141 (Theorem 5.4.2). □

1.3 Hausdorff Measure and Dimension

1.3.1 Hausdorff Measure

Recall Carathéodory's δ-approximative outer measure C_δ from (1.1.1). Carathéodory had noticed that his measure could be adjusted to give an m-dimensional measure in $\mathbb{R}^d$ for any $m \in \mathbb{Z}^+$ with $m < n$ as follows:

$$\mathcal{C}^m(E) = \lim_{\delta\to 0} \inf \left\{ \sum_{i=1}^{\infty} \text{diam}_m(E_i) \;\middle|\; E \subseteq \bigcup_{i=1}^{\infty} E_i \,,\ \text{diam}(E_i) < \delta \right\}$$

Here $\text{diam}_m(E_i)$ denotes the supremum of the m-dimensional volumes of all orthogonal projections of the convex hull of E_i onto all m-dimensional subspaces of $\mathbb{R}^d$.

Based on this, in 1918, almost thirty years after his graduation from Leipzig University which included a seven year hiatus from mathematical research proper, Felix Hausdorff produced a paper [Hau18] entitled *"Dimension und äußeres Maß"* or "Dimension and Outer Measure", which contained a brilliant insight. Hausdorff himself played down the importance of this insight by referring to it as a "kleinen Beitrag" or "small contribution" on top of Carathéodory's measure theory, but as it turned out, his discovery became the axle around which subsequent work in fractal geometry has revolved.

Hausdorff extended Carathéodory's m-dimensional measure so that it is based on summing the diameters of the E_i sets to the m-th power, i.e. using the following sum in the $\mathcal{C}^m$ definition:

$$\sum_{i=1}^{\infty} \text{diam}(E_i)^m.$$

He then noticed that this not only worked well when m is an integer, but also when m is any arbitrary real number. This small observation paved the way for the concept of non-integral dimension.

As Hausdorff observed, using this more liberal notion of dimension, for every set E there exists a unique critical value for m where the m-dimensional measure of E leaps between zero and infinity. This critical value is the Hausdorff dimension of E. Moreover, the measure of the E using this critical dimension value may be zero, finite or infinite.

It is worth noting here that Hausdorff dimension is sometimes referred to as Hausdorff-Besicovitch dimension, owing to the early work that Abraham Samilovitch Besicovitch contributed to the calculation of dimensions of fractal sets. For example, in [Bes35], [Bes34] and [BT54], Besicovitch *et al* compute the Hausdorff dimension of certain subsets of the line.

Definition 1.3.1. Let s be a non-negative real number. The *δ-approximative s-dimensional Hausdorff measure* $\mathcal{H}^s_\delta$ of a set $E \subseteq \mathbb{R}^d$ is defined as follows:

$$\mathcal{H}^s_\delta(E) = \inf\left\{ \sum_{i=1}^{\infty} \text{diam}(E_i)^s \;\middle|\; E \subseteq \bigcup_{i=1}^{\infty} E_i\,,\ \text{diam}(E_i) < \delta \right\}$$

In a similar way to the Carathéodory measure, as δ decreases, the class of permissible covers of E gets smaller and the δ-approximative measure approaches a limit value which we define as follows:

Definition 1.3.2. The *s-dimensional Hausdorff measure*:

$$\begin{aligned}\mathcal{H}^s(E) &= \lim_{\delta\to 0} \mathcal{H}^s_\delta(E) \\ &= \lim_{\delta\to 0} \inf\left\{ \sum_{i=1}^{\infty} \text{diam}(E_i)^s \;\middle|\; E \subseteq \bigcup_{i=1}^{\infty} E_i\,,\ \text{diam}(E_i) < \delta \right\}\end{aligned}$$

Theorem 1.3.3. *$\mathcal{H}^s$ is a measure on the Borel σ-algebra.*

Proof. Taking $\mathcal{M}_\delta$ to be the family of Borel sets in $\mathbb{R}^d$ with diameter less than δ and defining $\mathcal{T}_\delta(M_i)$ as $\text{diam}^s(M_i)$ where $M_i \in \mathcal{M}_\delta$, then using Definitions 1.2.1 and 1.2.6, H^s_δ is clearly the Method I outer measure associated with $(\mathcal{M}_\delta, \mathcal{T}_\delta)$ and $\mathcal{H}^s$ is its subsequent Method II outer measure. Theorem 1.2.7 states that all Method II outer measures are metric and hence, by Theorem 1.1.12, are measures on the Borel σ-algebra, so $\mathcal{H}^s$ is such a measure. □

A key property of Hausdorff measure, and indeed a property that we shall be making use of later on, is the scaling property.

Proposition 1.3.4. (Scaling Property of Hausdorff Measure)

If $F \subset \mathbb{R}^d$ and $\lambda > 0$ then

$$\mathcal{H}^s(\lambda F) = \lambda^s \mathcal{H}^s(F)$$

where $\lambda F = \{\lambda x : x \in F\}$, i.e. the set F scaled by a factor λ.

Proof. This proof can be found in [Fal90]. □

Definition 1.3.5. We call a Borel set with finite s-dimensional Hausdorff measure an s-set.

1.3.2 Hausdorff Dimension

Given a set F and some $\delta < 1$, and looking at the definition of $\mathcal{H}^s_\delta$ where we take the smallest sum of the diameters of covering sets to the s-th power, it is clear that $\mathcal{H}^s(F)$ is non-increasing as s increases. A more precise claim can be made when we analyse the situation a little more closely. Letting $t > s$ and $\bigcup_{i=1}^n U_i$ be a δ-cover of F, we have

$$\sum_{i=1}^n |U_i|^t \;\leq\; n\delta^t \;=\; n\delta^{t-s}\delta^s \;\leq\; \delta^{t-s}\sum_{i=1}^n |U_i|^s.$$

Therefore by taking infima on both sides, $\mathcal{H}^t_\delta(F) \leq \delta^{t-s}\mathcal{H}^s_\delta(F)$. If we let $\delta \to 0$, $\mathcal{H}^t(F)$ must be zero when $\mathcal{H}^s(F) < \infty$. We can see that as the value of s increases, $\mathcal{H}^s(F)$ tends closer to a critical value where it jumps from ∞ to 0. This critical value is the Hausdorff dimension of F. A formal definition follows.

Definition 1.3.6. The Hausdorff dimension $\dim_H$ of a non-empty set F is defined as follows:

$$\dim_H F \;=\; \inf\{s \mid \mathcal{H}^s(F) = 0\} \;=\; \sup\{s \mid \mathcal{H}^s(F) = \infty\}$$

so that

$$\mathcal{H}^s(F) = \begin{cases} \infty & \text{if } \ s < \dim_H F \\ 0 & \text{if } \ s > \dim_H F. \end{cases}$$

Some properties of Hausdorff dimension follow. Justification for these properties may be found in [Fal90].

(i) If $F \subset \mathbb{R}^d$ is open and non-empty, then $\dim_H F = d$.

(ii) If F is a continuously differentiable m-dimensional submanifold of $\mathbb{R}^d$, for instance a curve in $\mathbb{R}^2$ or a surface in $\mathbb{R}^3$, then $\dim_H F = m$.

(iii) If $E \subset F$, then $\dim_H E \leq \dim_H F$.

(iv) If $F_1, F_2, \ldots$ is a countable sequence of sets, then

$$\dim_H \bigcup_{i=1}^{n} F_i = \sup_{1 \leq i < \infty} \{\dim_H F_i\}.$$

(v) If F is countable, then $\dim_H F = 0$.

One serious disadvantage of the Hausdorff measure is that it can be difficult to calculate. We discuss techniques for accomplishing this in Section 1.5.

1.4 Box-Counting Dimension

Although we will not be making too much use of the Box-Counting dimension in the sequel, it is certainly helpful when computing the Hausdorff dimension and given its more practical usage relative to the Hausdorff dimension generally, it is certainly worth discussing here. While the Hausdorff dimension focuses *summing* the diameters of covering sets with diameter less than δ, the box-counting dimension involves *counting* the smallest number of sets of diameter less than δ that form a cover of the set being analysed. Computationally, the box-counting dimension is convenient since, as we shall see, it can be determined by coverings of sets of equal size and in many cases, it can be estimated as the gradient of a log-log graph plotted over a suitable range of δ.

On the other hand, it is not nearly as mathematically robust as the Hausdorff dimension, namely because it equates the dimension of a given set F to the dimension of its closure $\overline{F}$, the smallest closed subset of $\mathbb{R}^d$ which contains F. This means that it is possible for countable sets to have non-zero box counting dimension. For example, if we let $F = \{p \in \mathbb{Q} \cap [0,1]\}$, then $\overline{F} = [0,1]$ and therefore, $\underline{\dim}_B F = \overline{\dim}_B F = 1$.

1.4.1 Description

Definition 1.4.1. Let F be any non-empty bounded subset of $\mathbb{R}^d$ and let $N_\delta(F)$ be the smallest number of sets of diameter at most δ which can cover F. The *lower* and *upper box-counting dimensions* of F are

$$\underline{\dim}_B F = \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

and

$$\overline{\dim}_B F = \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

respectively. When these are equal, we refer to

$$\dim_B F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

as the *box-counting dimension* of F. As noted in [Fal90], in practice we may substitute the above definition of $N_\delta(F)$ for alternative definitions depending on the application environment, including but not limited to, any of the following:

(i) the smallest number of closed balls of radius δ that cover F;

(ii) the smallest number of cubes of side δ that cover F;

(iii) the number of δ-mesh cubes that intersect F;

(iv) the smallest number of sets of diameter at most δ that cover F;

(v) the largest number of disjoint balls of radius δ with centres in F.

The box counting dimension has been calculated for many of the fractal sets we see in the literature today. The calculation usually involves using definitions (i), (ii) or (iv) of $N_\delta(F)$ to determine $\overline{\dim}_B F$, then using definition (v) to find $\underline{\dim}_B F$, and checking to see whether these upper and lower bounds of $\dim_B F$ coincide.

1.4.2 Sample Calculation

We will demonstrate the box-counting dimension calculation for a classical simple fractal called the middle-third Cantor set. The middle-third Cantor set C is constructed by taking the unit interval $C_0 \subseteq \mathbb{R}^2$, removing the middle-third interval $(\frac{1}{3}, \frac{2}{3})$ and labeling the remainder C_1, then removing the middle-third intervals $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$ from the two remaining intervals in C_1 and labelling the subsequent remaining set C_2, and so on ad infinitum until we have $C = \bigcap_{k \in \mathbb{N}} C_k$.

Proposition 1.4.2. *Let C be the middle-third Cantor set constructed as described above.*

Then

$$\dim_B C = \underline{\dim}_B C = \overline{\dim}_B C = \frac{\log 2}{\log 3}.$$

Proof.

We start with the upper bound, $\overline{\dim}_B C$:

"$\leq$" Given C_k and $0 < \delta \leq 1$ such that $3^{-k} \leq \delta \leq 3^{-k+1}$, we may cover C_k with intervals of length 3^{-k} so that $N_\delta(C) \leq 2^k$.

Since $\delta \geq 3^{-k}$, more 3^{-k}-covers are required to cover C than δ-covers, so

$$\log N_\delta(C) \leq \log N_{3^{-k}}(C).$$

Also, since $\delta \leq 3^{-k+1}$, $-\log \delta \leq \log 3^{-k+1}$, so we get

$$\begin{aligned}\frac{\log N_\delta(C)}{-\log \delta} &\leq \frac{\log N_{3^{-k}}(C)}{-\log 3^{-k+1}} \\ &= \frac{\log 2^k}{\log 3^{k-1}} \\ &= \frac{\log 2^k}{\log 3^k + \log \frac{3^{k-1}}{3^k}}.\end{aligned}$$

Taking limits we get

$$\begin{aligned}\overline{\lim_{\delta \to 0}} \frac{\log N_\delta(C)}{-\log \delta} &\leq \overline{\lim_{k \to \infty}} \frac{\log 2^k}{\log 3^k} \\ &= \frac{\log 2}{\log 3}.\end{aligned}$$

"$\geq$" Any interval of length $\delta < 3^{-k}$ intersects at most one of the basic intervals at the kth level of the construction of C. There are 2^k such intervals at the kth level, so at least 2^k intervals of length δ are needed to cover C. Therefore

$$N_\delta(C) \geq 2^k \text{ if } \delta < 3^{-k}$$

so

$$\underline{\dim}_B C = \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(C)}{-\log \delta} \geq \underline{\lim}_{k \to \infty} \frac{\log 2^k}{\log 3^k} = \frac{\log 2}{\log 3}$$

follows in a similar way to the upper bound. □

1.4.3 Comparison with Hausdorff Dimension

Box-counting dimension is very useful when studying the Hausdorff dimension because it provides quite a useful upper bound for $\dim_H F$. We shall discuss this usefulness further in the next section, but for now we give the following result.

Proposition 1.4.3. *Let F be a subset of $\mathbb{R}^d$. If $\mathcal{H}^s(F) \geq 1$ and $s = \dim_H F$, then*

$$\dim_H F \leq \underline{\dim}_B F \leq \overline{\dim}_B F.$$

Proof. We can cover F with $N_\delta(F)$ sets of diameter δ. Thus,

$$\mathcal{H}^s_\delta(F) \leq N_\delta(F)\delta^s.$$

As $\delta \to 0$, $N_\delta(F)\delta^s \geq 1$ if δ is small enough. Taking logarithms of both sides we have

$$\log N_\delta(F) + s \log \delta \geq 0.$$

Adjusting the inequality we get

$$s \geq \frac{\log N_\delta(F)}{-\log \delta}$$

and taking the lower limit as $\delta \to 0$,

$$s \geq \varliminf_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}.$$

Therefore,

$$\dim_H F \leq \underline{\dim}_B F \leq \overline{\dim}_B F.$$

□

With a little more work, it is possible to prove a stronger version of the above theorem which says that $\dim_H F \leq \underline{\dim}_B F \leq \overline{\dim}_B F$ for all $F \subseteq \mathbb{R}^d$ regardless of the Hausdorff measure of F.

1.5 Techniques for Calculating Hausdorff Dimension

The main agenda of this dissertation is to discuss the calculation of the Hausdorff measure for certain popular fractal sets. To accomplish this, we will need to know the Hausdorff dimension of these sets and in this section, we illustrate how this can be calculated. As it happens, there is a convenient method for calculating the Hausdorff dimension of the particular types of fractals we will be looking at, namely self-similar sets, and we shall be examining this in the next chapter. For now we discuss a more general approach for calculating the upper and lower bounds of $\dim_H F$ for some $F \subset \mathbb{R}^d$.

1.5.1 Upper Bounds

As was previously illustrated, the box counting dimension of a set usually forms a good upper bound for its Hausdorff dimension. In the case of many fractal sets, it coincides conveniently with the lower bound when a good lower bound is found. Unfortunately, it is usually quite difficult to directly calculate a good lower bound for the Hausdorff dimension of most fractals, even in the simplest cases.

1.5.2 Lower Bounds and the Mass Distribution Principle

Finding a lower bound for the Hausdorff dimension without the aid of some helpful mathematical machinery is a troublesome task and one that often requires much rigorous work. Thankfully, such mathematical machinery is available in the guise of the *mass distribution principle* which we shall discuss momentarily.

Definition 1.5.1. Given a measure μ on $\mathbb{R}^d$, we refer to the smallest closed set X such that $\mu(\mathbb{R}^d \setminus X) = 0$ as the *support* of μ. We may also say that μ is a measure *supported on* the set A if A contains the support of μ.

Definition 1.5.2. We refer to a measure μ on a bounded subset of $\mathbb{R}^d$ as a *mass distribution* when $0 < \mu(\mathbb{R}^d) < \infty$. $\mu(A)$ may be thought of as the *mass* of a set A.

A mass distribution is usually constructed by spreading a finite mass in some obvious way over a set X. The way in which the mass is spread across the set usually depends on the construction of X itself. As an example of how a mass distribution might be used, consider the middle-third Cantor set C described in Section 1.4.2. If we assign a mass of say $\sqrt{2}$ to C_0, we then divide that mass evenly between the sets in C_1 so that each set gets mass $\frac{\sqrt{2}}{2}$. Each set in C_2 is given $\frac{1}{2}$ the mass of its parent set, i.e. $\frac{\sqrt{2}}{2.2}$, and so on for each level of the construction of C. The total mass being distributed is the same at each level of the construction.

The mass distribution principle helps us by allowing us to restrict the various component sets $\{U_i\}$ of a covering set of F so that no U_i covers too much of F relative to its own size, measured as $|U_i|^s$. This allows us to get an accurate estimate of the most efficient covering set for F.

Theorem 1.5.3. (mass distribution principle) *Let $F \subseteq \mathbb{R}^d$. Let us assume that we have a measure μ and two numbers, $c > 0$ and $\delta > 0$, such that*

1. $\mu(F) > 0$.

2. $\mu(U) \leq c|U|^s$ *for all* $U \subseteq \mathbb{R}^d$ *with* $|U| < \delta$.

Then

$$\mathcal{H}^s(F) \geq \frac{\mu(F)}{c} > 0.$$

In particular,

$$s \leq \dim_H F \leq \underline{\dim}_B F \leq \overline{\dim}_B F.$$

Proof. This proof can be found in [Fal90], but it is quite straightforward so we repeat it here for completeness. If $\{U_i\}$ is any δ-cover of F then

$$0 < \mu(F) \leq \mu\left(\bigcup_i U_i\right) \leq \sum_i \mu(U_i) \leq c \sum_i |U_i|^s.$$

Taking infima, $\mathcal{H}^s_\delta(F) \geq \frac{\mu(F)}{c}$ and so $\mathcal{H}^s(F) \geq \frac{\mu(F)}{c}$ as $\delta \to 0$. □

Chapter 2

Iterated Function Systems and Self-Similar Sets

2.1 Introduction

Many of the fractal sets discussed in the literature, and indeed the sets that we analyse in the research component of this dissertation, are *self-similar sets*, that is sets that are composed of smaller sets which are similar to the whole set. *Iterated Function Systems* or *IFSs* are families of mappings which may be used to generate such fractal sets based on their self-similar properties. Iterated Function Systems are extremely useful to us, not only because they provide a simple way to describe many fractal sets, but also because they are often instrumental in the calculation of both their measure and dimension.

In this chapter we provide a formal definition for IFSs and state one of the key results for them which will be used later when we calculate the Hausdorff measure of some Sierpinski fractals. We then prove the analogue of this result for a special breed of iterated function systems known as iterated function systems with condensation. We begin by discussing metric spaces.

2.2 Background Definitions and Theorems

2.2.1 Metric Spaces

Definition 2.2.1. A *metric space* is a pair (M, d) where M is a set and d is a map $d : M \times M \to \mathbb{R}$, such that

(i) $d(x, y) \geq 0$

(ii) $d(x, y) = 0 \Leftrightarrow x = y$ for all $x, y \in M$

(iii) $d(x, y) = d(y, x)$ for all $x, y \in M$

(iv) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in M$ (triangle inequality)

Definition 2.2.2. A sequence $(x_n)_n$ in M is called a *Cauchy sequence* if for all $\epsilon > 0$, there exists some number N such that for all $n, m \geq N$, $d(x_n, x_m) \leq \epsilon$.

Theorem 2.2.3. *(Cauchy criterion for convergence). A necessary and sufficient condition for convergence of a sequence $\{x_n\}$ is that it be a Cauchy sequence.*

Proof. A proof may be found in [Sut75] on pages 9-10. □

As is clear from the above theorem, we may prove that a sequence in $\mathbb{R}^n$ converges simply by proving that it is a Cauchy sequence, however, it is not in general true that all Cauchy sequences in a metric space converge. For instance, if a metric space is composed of all rational numbers with the metric $d(a, b) = |a - b|$, then a Cauchy sequence in that metric space may converge to an irrational number. So a Cauchy sequence in this metric space of irrational numbers may not converge to a limit in that space. Since the convergence of Cauchy sequences is important to us, we may proceed by defining a notion of *completeness* as follows.

Definition 2.2.4. A metric space is *complete* if all Cauchy sequences in the metric space converge.

2.2.2 Dynamical Systems and Banach's Contraction Principle

The basic notion of a contraction mapping in a metric space as defined next, forms the basis for Banach's contraction mapping theorem, an important theorem which is required in the next section on iterated function systems.

Definition 2.2.5. Let D be a metric space in $\mathbb{R}^n$. A mapping $S : D \to D$ is called a *contraction mapping* if there is a real number $0 \leq c < 1$ such that

$$d(S(x), S(y)) \leq c\, d(x, y)$$

for all x, y in D.

A contraction mapping is a specific type of a more general mapping known as a *Lipschitz mapping* where the contraction ratio c may be greater than 1. In the general case, $c \geq 0$ is referred to as the *Lipschitz constant* of a given Lipschitz mapping S or Lip(S). In the above definition, $0 \leq c < 1$ may be referred to as the *contraction ratio* of contraction mapping S.

Definition 2.2.6. We call the mapping $S : D \to D$ in the above definition a *similarity mapping* if we have

$$d(S(x), S(y)) = c\, d(x, y)$$

for all x, y in D.

The constant c may be referred to as a *similarity ratio* in the above definition.

The development of Banach's contraction theorem requires some concepts from dynamical systems theory. A dynamical system is a sequence whose terms are defined by repeatedly applying a mapping to some initial point. If the sequence converges to some point w, then w is called a fixed point of the system. The formal definitions for these two concepts are as follows:

Definition 2.2.7. Let D be a subset of $\mathbb{R}^n$ and let $f : D \to D$ be a continuous mapping, where f^k denotes the kth iterate of f, i.e. $f^0(x) = x, f^1(x) = f(x), f^2(x) = f(f(x))$ and so on. $\{f^k\}$ is called a *discrete dynamical system*.

Definition 2.2.8. Given a dynamical system $\{f^k\}$ in $D \subseteq \mathbb{R}^n$, if $f^k(x)$ converges to a point $w \in D$ where $f(w) = w$, then w is known as a *fixed point* of the dynamical system.

Now we may present Banach's contraction mapping theorem. This tells us that if we have a contraction mapping in a complete metric space, then there is a unique fixed point associated with this mapping and a dynamical system constructed using this mapping will converge to the fixed point no matter what initial point x we choose.

Theorem 2.2.9. *(Banach's contraction mapping theorem) Let (M, d) be a complete metric space. Let $S : M \to M$ be a contraction mapping. Then*

1. *S has a unique fixed point $p \in M$, such that $S(p) = p$.*

2. *$S^k(x) \to p$ as $k \to \infty$ for all $x \in M$.*

Proof. Let $x_1 \in M$ and $x_{k+1} = S(x_k)$, $k \in \mathbb{Z}_+$. So $x_{k+1} = S^k(x)$. We would like to show that $\{x_k\}$ is a Cauchy sequence. It is clear that

$$d(x_2, x_3) = d(S(x_2), S(x_3)) \leq cd(x_2, x_3)$$

for some constant $0 < c < 1$ by definition of a contraction. This implies that

$$d(x_3, x_4) = d(S(x_2), S(x_3)) \leq cd(x_2, x_3) \leq c^2 d(x_1, x_2).$$

Thus, $d(x_k, x_{k+1}) \leq c^{k-1} d(x_1, x_2)$ for $k \in \mathbb{Z}_+$.

Let m, n be any positive integers with $m > n$. By property (iii) of a metric space,

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m).$$

By the previous inequality,

$$\begin{aligned} d(x_n, x_m) &\leq (c^{n-1} + c^n + \cdots + c^{m-2}) d(x_1, x_2) \\ &= c^{n-1}(1 + k + \cdots + k^{m-n-1}) d(x_1, x_2) \end{aligned}$$

We have a geometric series on the right hand side, so

$$d(x_n, x_m) < c^{n-1} \left(\frac{1}{1-c} \right) d(x_1, x_2).$$

Fix $\epsilon > 0$. Since $0 < c < 1$, the right hand side of the above equation converges to 0 as $n \to \infty$, so there must exist a number N large enough such that for all $n > N$,

$$c^{n-1} \left(\frac{1}{1-c} \right) d(x_1, x_2) < \epsilon.$$

This implies that there also exists an N large enough where for all $n, m \geq N$ we have

$$d(x_n, x_m) < c^{n-1} \left(\frac{1}{1-c} \right) d(x_1, x_2) \leq \epsilon,$$

thus showing that $\{x_k\}$ is a Cauchy sequence. We know that S is continuous because for all $x, y \in M$, $d(S(x), S(y)) \leq cd(x, y)$. So if we take p to be the limit of $\{x_k\}$, $S(x_k) \rightarrow S(p)$ as $k \rightarrow \infty$. Since $x_{k+1} \rightarrow p$ and $x_{k+1} = S(x_k)$, $S(x_k) \rightarrow p$, so $S(p) = p$.

To prove the uniqueness of the fixed point, let $a, b \in M$ both be fixed points of S. Thus, $d(S(a), S(b)) \leq cd(a, b)$. Since $0 < c < 1$ and $d(S(a), S(b)) = d(a, b)$, $d(a, b) = 0$, thus $a = b$.

□

2.3 Iterated Function Systems

2.3.1 Basic Definition

As was discussed at the beginning of this chapter, iterated function systems are very important to us as they are indelibly linked to techniques used for calculating the Hausdorff measure of fractals which we will be discussing in subsequent chapters. They were dissected in John E. Hutchinson's seminal 1981 paper [Hut81] and further explored in the book "Fractals Everywhere" by Michael F. Barnsley [Bar88] in 1988. Many of the results contained in Hutchinson's paper were also derived in an earlier work by P.A.P. Moran entitled *Additive functions of intervals and Hausdorff measure* [Mor46]. Here we provide the basic definition of an iterated function system or *IFS*:

Definition 2.3.1. Let D be a closed subset of $\mathbb{R}^n$. Let $(S_1, \ldots, S_m)$ be contractions on D such that

$$|S_i(x) - S_i(y)| \leq r_i|x - y|$$

for all x, y in D where the r_i are contraction ratios such that $0 < r_i < 1$. The collection of mappings $\{S_1, \ldots, S_m\}$ is called an *iterated function system* or *IFS*.

There is a key result for iterated function systems that shows that IFSs have a unique attractor or invariant set associated with them. Moreover, this result also shows that if an IFS is applied to any non-empty compact subset of the space that it is acting on (usually $D \subset \mathbb{R}^n$ or $\mathbb{R}^n$ itself), then applied to the resulting set and this process is repeated infinitely many times, the resultant set will be the invariant set associated with the IFS. This remarkable result due to Hutchinson [Hut81], is a special case of Banach's contraction mapping theorem, except that instead of having infinitely many iterations of a *single* contraction mapping acting on a point and converging to a fixed point, we have infinitely many iterations of a *family* of contractions acting on a set and converging to an invariant set. The result is formalised as follows:

Theorem 2.3.2. *Let D be a closed non-empty subset of $\mathbb{R}^n$ and let the family of contractions $\{S_1, \ldots, S_m\}$ be an IFS acting on D. Let $\mathcal{S}$ denote the family of all non-empty, compact subsets of D.*

(i) There exists a non-empty compact invariant set $F \subseteq D$, such that

$$F = \bigcup_{i=1}^{m} S_i(F).$$

(ii) If we define $S : \mathcal{S} \to \mathcal{S}$ to be $S(E) = \bigcup_{i=1}^{m} S_i(E)$ for $E \in \mathcal{S}$ and write S^k for the kth iterate of S so that $S^0(E) = E$ and $S^k(E) = S(S^{k-1}(E))$ for $k \geq 1$, then

$$F = \bigcap_{k=0}^{\infty} S^k(E)$$

for every set $E \in \mathcal{S}$ such that $S_i(E) \subset E$ for all i.

There are two different well-known techniques for proving this result, one of which is a set theoretical method, the the other of which relies on Banach's contraction mapping theorem and is perhaps a bit more elegant. Discussions may be found in [Fal90]. Neither method is explored here for regular IFSs, but there exists a special type of IFS which we will be discussing next and for which we will prove the analogue of the above theorem using the Banach contraction mapping theorem technique.

2.3.2 Iterated Function Systems with Condensation

During the course of his studies, the author took a particular interest in iterated function systems with condensation. These special types of IFS, which were introduced by Barnsley in his book [Bar88], work by adding a non-empty, compact set called a "condensation set" to each level of the construction of a given IFS. In practice, this allows for the invariant sets of two different IFSs to be mixed together in some way, thus expanding the class of sets which may be produced using IFS techniques. An example of the construction of an IFS with condensation may be seen in Figure 6.1.1. Further discussion on different constructions of IFSs with condensation and on algorithms used to generate figures of their respective invariant sets may be found in a book by Mario Peruggia, "Discrete Iterated Function Systems" [Per93]. We provide a formal definition next:

Definition 2.3.3. Let D be a closed non-empty subset of $\mathbb{R}^n$. Let $\mathcal{S}$ denote the family of all non-empty, compact subsets of D. Let $(S_1, \ldots, S_m)$ be contractions on D such that

$$|S_i(x) - S_i(y)| \leq r_i|x - y|$$

for all x, y in D where the r_i are contraction ratios such that $0 < r_i < 1$. Choose a fixed, non-empty compact set $C \in \mathcal{S}$ and a mapping $S_0 : \mathcal{S} \to \mathcal{S}$, such that $S_0(B) = C$ for any $B \in \mathcal{S}$. The collection of mappings $\{S_0, \ldots, S_m\}$ is called an *iterated function system with condensation* or *IFS with condensation* where C is the associated condensation set.

In the following section we will prove the IFS with condensation analogue of Theorem

2.3.2, a result we will be making use of in Chapter 6. As it turns out, there is only a minimal difference between this proof and the proof of Theorem 2.3.2.

2.3.3 Existence and Uniqueness of Invariant Sets for IFSs with Condensation

Theorem 2.3.4. *Let D be a closed non-empty subset of $\mathbb{R}^n$ and let $\mathcal{S}$ denote the family of all non-empty, compact subsets of D. Let the family of mappings $\{S_0, \ldots, S_m\}$ be an IFS with condensation acting on D, where $C \in \mathcal{S}$ is the associated condensation set with $S_0(B) = C$ for all $B \in \mathcal{S}$. Let $\{r_1, \ldots, r_m\}$ be the contraction ratios for the contractions $\{S_1, \ldots, S_m\}$.*

(i) There exists a non-empty compact invariant set $F \subseteq D$, such that

$$F = \bigcup_{i=0}^{m} S_i(F) = C \cup \left(\bigcup_{i=1}^{m} S_i(F) \right).$$

(ii) If we define $S : \mathcal{S} \to \mathcal{S}$ to be $S(E) = \bigcup_{i=0}^{m} S_i(E)$ for $E \in \mathcal{S}$ and write S^k for the kth iterate of S so that $S^0(E) = E$ and $S^k(E) = S(S^{k-1}(E))$ for $k \geq 1$, then

$$F = \bigcap_{k=0}^{\infty} S^k(E)$$

for every set $E \in \mathcal{S}$ such that $S_i(E) \subset E$ for all i.

We require the following definition and subsequent lemmas before proceeding with the proof of Theorem 2.3.4:

Definition 2.3.5. We define the *Hausdorff metric* or *Hausdorff distance* between two sets

A and B as follows:

$$\begin{aligned} d(A,B) &= \max\left\{\sup_{a\in A}\inf_{b\in B}|a-b|,\ \sup_{b\in B}\inf_{a\in A}|a-b|\right\} \\ &= \max\left\{\inf\left\{\delta : A\subseteq B_\delta\right\},\ \inf\left\{\delta : B\subseteq A_\delta\right\}\right\} \\ &= \inf\left\{\delta : A\subseteq B_\delta,\ B\subseteq A_\delta\right\} \end{aligned}$$

where

$$A_\delta = \{x\in D : |x-a|\le\delta \text{ for some set } A \text{ where } a\in A\}$$

and

$$B_\delta = \{x\in D : |x-a|\le\delta \text{ for some set } A \text{ where } a\in B\}$$

i.e. A_δ and B_δ are δ-neighbourhoods of A and B respectively.

Lemma 2.3.6. *Let $\{S_1,\ldots,S_m\}$ be an IFS on some metric space D of $\mathbb{R}^n$. Let A and B be two non-empty compact subsets of D. Then*

$$d\left(\bigcup_{i=1}^m S_i(A),\bigcup_{i=1}^m S_i(B)\right)\le\max_{1\le i\le m} d\left(S_i(A),S_i(B)\right).$$

Proof. It is sufficient to show that

$$d\left(\bigcup_{i=1}^m A_i,\bigcup_{i=1}^m B_i\right)\le\max_{1\le i\le m} d\left(A_i,B_i\right)$$

where $\{A_i\}_i$ and $\{B_i\}_i$ are collections of non-empty compact subsets of D.

$$\begin{aligned} d\left(\bigcup_{i=1}^m A_i,\bigcup_{i=1}^m B_i\right) &= \max\left\{\sup_{a\in\bigcup_{i=1}^m A_i}\ \inf_{b\in\bigcup_{i=1}^m B_i}|a-b|,\ \sup_{b\in\bigcup_{i=1}^m B_i}\ \inf_{a\in\bigcup_{i=1}^m A_i}|a-b|\right\} \\ &\le \max\left\{\sup_{0\le i\le m}\sup_{a\in A_i}\inf_{b\in B_i}|a-b|,\ \sup_{0\le i\le m}\sup_{b\in B_i}\inf_{a\in A_i}|a-b|\right\} \end{aligned}$$

$$= \sup_{0\le i\le m} \max\left\{\sup_{a\in A_i}\inf_{b\in B_i}|a-b|,\ \sup_{b\in B_i}\inf_{a\in A_i}|a-b|\right\}$$
$$= \sup_{0\le i\le m} d(A_i, B_i).$$

□

Lemma 2.3.7. *Let A and B be two non-empty compact subsets of a metric space D in $\mathbb{R}^n$ and let f be a contraction mapping. Then*

$$d\left(f(A), f(B)\right) \le \mathit{Lip}(f) d\left(A, B\right).$$

Proof.

$$\begin{aligned} d\left(f(A), f(B)\right) &= \max\left\{\sup_{a\in A}\inf_{b\in B}|f(a)-f(b)|,\ \sup_{b\in B}\inf_{a\in A}|f(a)-f(b)|\right\} \\ &\le \max\left\{\sup_{a\in A}\inf_{b\in B}\mathrm{Lip}(f)|a-b|,\ \sup_{b\in B}\inf_{a\in A}\mathrm{Lip}(f)|a-b|\right\} \\ &= \mathrm{Lip}(f)\max\left\{\sup_{a\in A}\inf_{b\in B}|a-b|,\ \sup_{b\in B}\inf_{a\in A}|a-b|\right\} \\ &= \mathrm{Lip}(f) d(A, B). \end{aligned}$$

□

Lemma 2.3.8. *Let $S : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function. Then if $E \subseteq \mathbb{R}^n$ is compact, its image under S, is also compact.*

Proof. Since E is compact, given any sequence $\{y_i\}_{i=1}^{\infty} \in E$, there exists a convergent subsequence $\{y_j\}_{j=1}^{\infty}$ such that

$$\lim_{j\to 0} y_j = y \in E.$$

Using the Heine definition of continuity, since S is continuous,

$$\lim_{j\to\infty} S(y_j) = S(y) \in S(E).$$

Of course $\{S(y_i)\}_{i=1}^{\infty} \in S(E)$ and contains the subsequence $\{S(y_j)\}_{j=1}^{\infty}$, so S(E) is compact. $\square$

We may now prove Theorem 2.3.4.

Proof. First we define a suitable metric space $(\mathcal{S}, d)$ for non-empty compact subsets of D using the Hausdorff metric d between two such subsets A and B which is defined as follows:

It is easily seen that d satisfies the three requirements of a metric and one can show that $(\mathcal{S}, d)$ is a complete metric space, a proof of which may be found in [HS91] (pages 77-78). Let $A, B \in \mathcal{S}$. Then using Lemmas 2.3.6 and 2.3.7, we have

$$\begin{aligned} d(S(A), S(B)) &= d\left(\bigcup_{i=0}^{m} S_i(A), \bigcup_{i=0}^{m} S_i(B)\right) \\ &\leq \max_{1\leq i\leq m} d(S_i(A), S_i(B)) \\ &\leq \left(\max_{1\leq i\leq m} r_i\right) d(A, B). \end{aligned}$$

since $d(S_0(A), S_0(B)) = d(C, C) = 0$ for all $A, B \in \mathcal{S}$. Thus, S is a contraction on the complete metric space $(\mathcal{S}, d)$. By Banach's contraction mapping theorem, there exists a unique fixed point $F \in \mathcal{S}$ for S, i.e.

$$F = S(F) = \bigcup_{i=0}^{m} S_i(F) = C \cup \left(\bigcup_{i=1}^{m} S_i(F)\right).$$

This proves (i).

Since S is a contraction and $S_i(E) \subseteq E$ for all i, we have a decreasing sequence as follows

$$E \supseteq S(E) \supseteq S^2(E) \supseteq \cdots \supseteq \bigcap_{k=0}^{\infty} S^k(E) \tag{2.3.1}$$

for all $E \in \mathcal{S}$. The second part of Banach's contraction mapping theorem tells us that $S^k(E) \to F$ as $k \to \infty$. Since $S^k(K)$ is a decreasing sequence of sets and the sequence converges, then it must converge at the intersection of all the sets in the sequence, so

$$F = \bigcap_{k=0}^{\infty} S^k(E).$$

This proves (ii).

□

In the sequel, sometimes it will be necessary for us to refer to sequences of mappings from an IFS acting over other mappings from the IFS, so we proceed with the following small definition to ease the notational burden.

Definition 2.3.9. Let $\{S_1, \ldots, S_m\}$ be an IFS in $\mathbb{R}^n$. Then $S_{i_1 \ldots i_p} = S_{i_1} \circ \cdots \circ S_{i_p}$ where $i_j \in \{1, \ldots, m\}$ for all j.

2.4 Self-Similar Sets

2.4.1 Definition

We will now discuss a special type of invariant set called a *self-similar* set. As was mentioned, many common fractals in the literature are self-similar sets. The Cantor set, the Von Koch curve and the Sierpinski triangle are all examples of self-similar sets. These sets are constructed using mappings which do not alter the geometrical shape of sets they are acting on. The mappings simply re-scale sets by some scaling factor $0 < \lambda < 1$. A more general class of such sets called *self-affine* sets are based on affine transformations which contract with differing ratios in different directions. The fractals we analyse later in this dissertation are self-similar, so we will not be discussing the self-affine class of fractals here.

Hutchinson provides us with a formal definition for self-similar sets in [Hut81]:

Definition 2.4.1. Let D be a closed subset of $\mathbb{R}^n$ and let $\{S_1, \ldots, S_m\}$ be an IFS on D. Then we call a set K *self-similar* with respect to $\{S_1, \ldots, S_m\}$ if

(i) K is invariant with respect to $\{S_1, \ldots, S_m\}$ and

(ii) $\mathcal{H}^s(K) > 0, \mathcal{H}^s(S_i(K) \cap S_j(K)) = 0$ for $i \neq j$, where $s = \dim_H K$.

2.4.2 Dimensions of Self-Similar Sets

Calculating both the box-counting and the Hausdorff dimensions of self-similar sets is made relatively easy thanks to a very useful theorem. This theorem tells us that if we have a self-similar set F with similarity mappings $S_1, \ldots, S_m$ and contraction ratios $r_1, \ldots, r_m$, and if the $S_i(F)$ 'do not overlap too much', then F has equal box-counting and Hausdorff dimensions. As well as that the theorem gives us an easy way to compute this dimension value and F will have positive and finite Hausdorff measure, i.e. F will be an s-set. We will not prove this theorem here, though we will be making use of it later on so it is certainly worth noting. The proof requires a more concrete version of the 'do not overlap too much' requirement, known as the *open set condition*.

Definition 2.4.2. Given a self-similar set F based on similarities $S_1, \ldots, S_m$ and respective contraction ratios $r_1, \ldots, r_m$, we say that the S_i satisfy the *open set condition* if there exists a non-empty bounded open set V such that

$$\bigcup_{i=1}^{m} S_i(V) \subset V$$

where $S_i(V) \cap S_j(V) = \emptyset$ when $i \neq j$.

Theorem 2.4.3. *Let F be the self-similar set that results from the IFS $S_1, \ldots, S_m$ and let the open set condition hold for the S_i. Then $\dim_B F = \dim_H F = s$, where s is given by*

$$\sum_{i=1}^{m} r_i^s = 1.$$

Moreover, $0 < \mathcal{H}^s(F) < \infty$.

Proof. This proof may be found in [Fal90]. □

Chapter 3

Techniques for Calculating Hausdorff Measure

3.1 Introduction

As outlined in [Fal86] and [ZF04], the Hausdorff measure of a set at the critical dimension is notoriously difficult to calculate. While the notion of Hausdorff measure is convenient mathematically due to the fact that it is based on measure theory, finding general methods for its calculation for a wide class of sets has proven to be elusive. In [ZF04], on the problem of calculating the Hausdorff measure, Zhou and Feng reason that the difficulty is not one of “computational trickiness nor computational capacity, but a lack of full understanding of the essence of the Hausdorff measure”. A number of authors have attempted to calculate both the Hausdorff dimension and the Hausdorff measure of various popular fractal sets. In the following two sections of this chapter, Sections 3.2 and 3.3, we discuss the important relationship between the local density of fractal sets and Hausdorff measure, and in the last section, Section 3.4 we give a short review of attempts by various authors to calculate the Hausdorff dimension and measure of various sets.

3.2 Local Spherical Density and Hausdorff Measure

3.2.1 Discussion and Definitions

As is suggested in both [ZF04] and [AS99], the local density of a self-similar set which satisfies the open set condition is closely related to its Hausdorff measure and the main focus of this chapter is to mount a detailed investigation into this relationship. We require the following definition for our discussion of density:

Definition 3.2.1. A property is said to hold *almost everywhere* or for *almost all* $x \in E$ with respect to a measure μ if it holds for all $x \in E$ except for a set of μ-measure zero.

The local density of a set F at a point x can be thought of as an estimate of the level of concentration of points from F in the neighbourhood of x. One such estimate is Lebesgue's density. In order to formulate it, we need to know about Lebesgue measure:

Definition 3.2.2. If $A = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : a_i \leq x_i \leq b_i\}$ is a 'coordinate parallelpiped' and the n-dimensional volume of A is given by

$$\mathrm{vol}^n(A) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n),$$

we may define the *n-dimensional Lebesgue measure* $\mathcal{L}^n$ to be

$$\mathcal{L}^n(A) = \inf\left\{\sum_{i=1}^{\infty} \mathrm{vol}^n(A_i) : A \subset \bigcup_{i=1}^{\infty} A_i\right\}$$

where the infimum is taken over all coverings of A by coordinate parallelpipeds A_i. $\mathcal{L}^n$ may be shown to be a measure in $\mathbb{R}^n$.

Now that we know what a Lebesgue measure looks like, we may formulate Lebesgue's density as follows:

Definition 3.2.3. We refer to

$$D(F,x) = \lim_{r\to 0} \frac{\mathcal{L}^n(F \cap B_r(x))}{\mathcal{L}^n(B_r(x))}$$

as the *Lebesgue density* of a Borel set F in $\mathbb{R}^n$ if the limit exists.

A classical result known as Lebesgue's Density Theorem gives us some insight into when the limit does exist, as follows:

$$D(F,x) = 1 \text{ for } \mathcal{L}^n\text{-almost all } x \in F.$$

Unfortunately, this theorem is not so useful for fractal sets since $\mathcal{L}^n(F) = 0$ if $\dim(F) < n$, so the obvious approach in this situation is to reformulate density so that it uses a measure which can cope with non-integral dimensions, i.e. the Hausdorff measure. Fortunately, it is possible to reformulate density in such a way and achieve positive results. In the following definition recall that an s*-set* is a Borel set of Hausdorff dimension s with positive finite s-dimensional Hausdorff measure:

Definition 3.2.4. The *lower* and *upper densities* of an s-set F at a point $x \in \mathbb{R}^n$ are defined as

$$\underline{D}^s(F,x) = \underline{\lim}_{r\to 0} \frac{\mathcal{H}^s(F \cap B_r(x))}{(2r)^s} \quad \text{and} \quad \overline{D}^s(F,x) = \overline{\lim}_{r\to 0} \frac{\mathcal{H}^s(F \cap B_r(x))}{(2r)^s}$$

respectively.

Note: Hereafter, we may refer to the lower density as *lower spherical density* and upper density as *upper spherical density* interchangeably.

Work on the local density of sets was championed by Besicovitch in the 1920's and 30's, resulting in the three seminal papers [Bes28], [Bes38] and [Bes39]. He formulated a density boundedness theorem which directly relates Hausdorff measure to upper density for sets with finite s-dimensional Hausdorff measure, a result which we present here as Theorem 3.2.6 in Section 3.2.3. The theorem says that given an s-set F in $\mathbb{R}^d$, $2^{-s} \leq \overline{D}^s(F,x) \leq 1$ for $\mathcal{H}^s$-almost all $x \in F$.

A key observation is that the upper s-dimensional density is not as useful with respect to the s-dimensional Hausdorff measure as it could be. This is due to the fact that the upper s-dimensional density, as we have defined it above, bases its estimate of the density of a set at a point x on strictly spherical sets, whereas the Hausdorff measure uses a more liberal policy with its covering sets. For this reason, we introduce the upper s-dimensional convex density in Section 3.3 which uses open convex sets instead of balls for its estimates of local density and allows us to garner some very useful results with respect to the Hausdorff measure. There is a variation on the usual Hausdorff measure called the s-dimensional spherical Hausdorff measure which forms a more suitable accompaniment to the s-dimensional density. Some of the results involving convex density and the usual Hausdorff measure have analogues for the spherical density and spherical Hausdorff measure definitions. For example, when upper s-dimensional spherical density is reformulated to use spherical Hausdorff measure, the result $\overline{D}^s(F,x) = 1$ for $\mathcal{H}^s_{\mathcal{S}}$-almost all x in an s-set $F \subseteq \mathbb{R}^d$ may be acquired, where

$$\mathcal{H}^s_{\mathcal{S}}(E) = \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} \text{diam}(E_i)^s \;\middle|\; E \subseteq \bigcup_{i=1}^{\infty} E_i\,,\ \text{diam}(E_i) < \delta, E_i \text{ is a ball} \right\}.$$

Discussions may be found in [Mat95] and [Ols05].

3.2.2 A Background Result

We require the following result in Section 3.2.3 for our discussion of Besicovitch's density boundedness result, Theorem 3.2.6.

Proposition 3.2.5. *Let μ be a mass distribution on $\mathbb{R}^n$, let $F \subset \mathbb{R}^n$ be a Borel set and let $0 < c < \infty$ be a constant.*

(i) If $\overline{\lim_{r\to 0}} \dfrac{\mu(B_r(x))}{r^s} < c$ *for all* $x \in F$*, then*

$$\mathcal{H}^s(F) \geq \frac{\mu(F)}{c}.$$

(ii) If $\underline{\lim_{r\to 0}} \dfrac{\mu(B_r(x))}{r^s} > c$ *for all* $x \in F$*, then*

$$\mathcal{H}^s(F) \leq \frac{2^s\mu(\mathbb{R}^n)}{c}.$$

Proof.

(i) Let

$$F_\delta = \{x \in F : \mu(B_r(x)) < c\,r^s \text{ for all } 0 < r \leq \delta\}$$

for all $\delta > 0$ and observe that $F = \bigcup_{\delta\in\mathbb{Q}_+} F_\delta$. Let $\{U_i\}$ be a δ-cover of F. Then $F_\delta \subseteq \bigcup_i U_i$. Also, for all U_i where there exists an $x \in U_i$ such that $x \in F_\delta$, then $U_i \subseteq B_{|U_i|}(x)$. By definition of F_δ $\mu(U_i) \leq \mu(B_{|U_i|}(x)) < c|U_i|^s$ so

$$\begin{aligned} \mu(F_\delta) &\leq \mu\left(\bigcup_{i=1}^{\infty}\{U_i : U_i \cap F_\delta \neq \emptyset\}\right) && (3.2.1)\\ &\leq \sum_{i=1}^{\infty}\{\mu(U_i) : U_i \cap F_\delta \neq \emptyset\} && (3.2.2)\end{aligned}$$

$$< \sum_{i=1}^{\infty} c|U_i|^s. \tag{3.2.3}$$

Since $\{U_i\}$ is any δ-cover of F, we have $\mu(F_\delta) \leq c\mathcal{H}^s_\delta(F)$. As Falconer points out in [Fal90] (page 11), when $\delta > 0$ and we have Borel sets A_δ that are increasing as δ decreases, then $\lim_{\delta\to 0} \mu(A_\delta) = \mu\left(\bigcup_{\delta>0} A_\delta\right)$, so

$$\mu(F) = \mu\left(\bigcup_{\delta\in\mathbb{Q}_+} F_\delta\right) = \lim_{\delta\to 0} \mu(F_\delta) \leq \lim_{\delta\to 0} c\mathcal{H}^s_\delta(F) = c\mathcal{H}^s(F).$$

(ii) The proof of this is omitted but may be found in [Fal90].

□

3.2.3 Local Density Bounds

We now turn to the important theorem due to Besicovitch which relates the Hausdorff measure of s-sets to their local spherical density at certain points. The result states that the spherical density of a given s-set lies within a certain range $\mathcal{H}^s$-almost everywhere. In [Fal90], Falconer gives a shortened proof for the lower bound and states that the upper bound "follows in essentially the same way". We expand the proof for the lower bound here and show that the upper bound does not in fact follow quite so easily. It is not immediately obvious exactly how this result might be applied in calculating the Hausdorff measure of fractal sets; this is a problem we look at in Section 3.3.4.

Theorem 3.2.6. *Let F be an s-set in $\mathbb{R}^n$. Then $2^{-s} \leq \overline{D}^s(F,x) \leq 1$ for $\mathcal{H}^s$-almost all $x \in F$.*

Proof.

"$2^{-s} \leq \overline{D}^s(F,x)$"

Let $n \in \mathbb{N}$. Put $c_n = 1 - \frac{1}{n}$. Let $\mu(A) = \mathcal{H}^s(F \cap A)$. If

$$F_n = \left\{ x \in F : \overline{D}^s(F,x) = \overline{\lim_{r\to 0}} \frac{\mathcal{H}^s(F \cap B_r(x))}{(2r)^s} < 2^{-s} c_n \right\}$$

then we would like to show that $\overline{\lim_{r\to 0}} \dfrac{\mu(B_r(x))}{r^s} < c_n$ for all $x \in F_n$. For all $x \in F_n$ we have

$$\begin{aligned}
\overline{\lim_{r\to 0}} \frac{\mu(B_r(x))}{r^s} &= \overline{\lim_{r\to 0}} \frac{\mathcal{H}^s(F \cap B_r(x))}{r^s} \\
&= 2^s \overline{\lim_{r\to 0}} \frac{\mathcal{H}^s(F \cap B_r(x))}{(2r)^s} \\
&< (2^{-s} c_n).2^s \\
&= c_n.
\end{aligned}$$

This is true for all $x \in F_n$, so by Proposition 3.2.5 (i) we have

$$\mathcal{H}^s(F_n) \geq \frac{\mu(F_n)}{c_n} = \frac{\mathcal{H}^s(F_n)}{c_n}.$$

Since F is an s-set, and therefore F_n is also an s-set, we know that $\mathcal{H}^s(F_n)$ is positive and finite, so $c_n \mathcal{H}^s(F_n) \geq \mathcal{H}^s(F_n)$ implies that $\mathcal{H}^s(F_n) = 0$. We would like to show that $2^{-s} \leq \overline{D}^s(F,x)$ for $\mathcal{H}^s$-almost all $x \in F$, in other words, we would like to show that $\mathcal{H}^s\left(\left\{x \in F : 2^{-s} > \overline{D}^s(F,x)\right\}\right) = 0$. Clearly

$$\left\{x \in F : 2^{-s} > \overline{D}^s(F,x)\right\} = \bigcup_{n=1}^{\infty} \left\{x \in F : 2^{-s} c_n > \overline{D}^s(F,x)\right\} = \bigcup_{n=1}^{\infty} F_n$$

and obviously since these sets are equal, their Hausdorff measures coincide, so we have

$$\begin{aligned}
\mathcal{H}^s\left(\left\{x \in F : 2^{-s} > \overline{D}^s(F,x)\right\}\right) &= \mathcal{H}^s\left(\bigcup_{n=1}^{\infty} F_n\right) \\
&\leq \sum_{n=1}^{\infty} \mathcal{H}^s(F_n)
\end{aligned}$$

$$= \quad 0.$$

Thus $\overline{D}^s(F,x) \geq 2^{-s}$ holds for all points in F except for the set $\left\{x \in F : 2^{-s} > \overline{D}^s(F,x)\right\}$ of Hausdorff measure zero, i.e. $\overline{D}^s(F,x) \geq 2^{-s}$ holds for $\mathcal{H}^s$-almost all $x \in F$.

"$\overline{D}^s(F,x) \leq 1$"

We will set about proving this inequality in a similar way to the above using Proposition 3.2.5 (ii) as described in [Fal90], but as we shall see, the proof breaks down.

Again, let $n \in \mathbb{N}$ and let $\mu(A) = \mathcal{H}^s(F \cup A)$. Put $c_n = 2^s(1 + \frac{1}{n})$.

We would like to show that $\overline{D}^s(F,x) \leq 1$ for all points in F except for a set of $\mathcal{H}^s$-measure zero, that is $E = \left\{x \in F : \overline{D}^s(F,x) > 1\right\}$ and $\mathcal{H}^s(E) = 0$. Put

$$F_n = \left\{ x \in F : \overline{D}^s(F,x) = \varlimsup_{r \to 0} \frac{\mathcal{H}^s(F \cap B_r(x))}{(2r)^s} > 1 + \frac{1}{n} \right\}.$$

It suffices to show that F_n has zero-mass. For all $x \in F_n$ we have

$$\begin{aligned} \varlimsup_{r \to 0} \frac{\mu(B_r(x))}{r^s} &= \varlimsup_{r \to 0} \frac{\mathcal{H}^s(F \cap B_r(x))}{r^s} \\ &= 2^s \varlimsup_{r \to 0} \frac{\mathcal{H}^s(F \cap B_r(x))}{(2r)^s} \\ &> 2^s \left(1 + \frac{1}{n}\right) \\ &= c_n. \end{aligned}$$

Hence by Proposition 3.2.5 (ii),

$$\begin{aligned} \mathcal{H}^s(F_n) &\leq \frac{2^s \mu(\mathbb{R}^n)}{c_n} \\ &= \frac{2^s \mathcal{H}^s(\mathbb{R}^n \cap E)}{2^s(1 + \frac{1}{n})} \\ &= \frac{\mathcal{H}^s(E)}{\frac{n+1}{n}} \end{aligned}$$

$$= \frac{n}{n+1}\mathcal{H}^s(E).$$

The proof breaks down at this point. Had we had $\mathcal{H}^s(F_n) \leq \frac{n}{n+1}\mathcal{H}^s(F_n)$, instead of the above inequality, we could have shown that $\mathcal{H}^s(F_n) = 0$, thus finishing the proof. Unfortunately a more complicated method of proving $\overline{D}^s(F,x) \leq 1$ must be resorted to. This method is explored in [Fal86] and we will not be examining it here.

3.3 Local Convex Density and Hausdorff Measure

3.3.1 Discussion and Definitions

The type of density we present here is much more useful with regard to the Hausdorff measure than regular spherical density. In Section 3.3.2 we look at a Theorem which is analogous to Besicovitch's Theorem 3.2.6 for spherical density. It is however, a more precise result which helps gives rise to some more powerful results which we analyse is Section 3.3.4.

First, we present the relevant definition:

Definition 3.3.1. The *upper convex density* of an s-set F at a point $x \in \mathbb{R}^d$ is defined as

$$\overline{D}_c^s(F,x) = \overline{\lim_{r\to 0}}\left\{\sup \frac{\mathcal{H}^s(F\cap U)}{|U|^s}\right\}$$

where the supremum is over all open convex sets U with $x \in U$ and $0 < |U| < r$.

Later on in Section 3.3.3 we provide a new result, one of the implications of which is that given some s-set E in $\mathbb{R}^d$, $\overline{D}_c^s(E,x) \leq 1$ for all x. The key theorem in Section 3.3.2, Theorem 3.3.11, says that $\overline{D}_c^s(E,x) = 1$ for $\mathcal{H}^s$-almost all $x \in E$. As is pointed out in [ZF04] by Zhou and Feng, an obvious consequence of this is that the set

$E_0 = \{x \in E \mid \overline{D}_c^s(E,x) = 1\}$ is measurable and that $\mathcal{H}^s(E_0) = \mathcal{H}^s(E)$. One question that emerges is, under what conditions is $E_0 = E$? In the same paper, Zhou and Feng, provide an interesting discussion on upper convex density and pose some more interesting questions. Two of these questions are:

(i) Given an s-set E, under what conditions is there a set V with $x \in V$ such that $\overline{D}_c^s(E,x) = \dfrac{\mathcal{H}^s(E \cap V)}{|V|^s}$?

(ii) If such a set V exists, how does one determine its geometric shape or form?

Such questions have been tackled in the literature by various authors, e.g. [Mar86, Mar87, AS99], for a number of different fractal sets. We discuss this and related matters further in Sections 3.3.4 and 3.4.

We require the notion of a Vitali class and Vitali's covering theorem to prove a result in Section 3.3.2:

Definition 3.3.2. A collection of sets $\mathcal{V}$ is called a *Vitali class* for F if for each $x \in F$ and $\delta > 0$ there exists $U \in \mathcal{V}$ with $x \in U$ and $0 < |U| \leq \delta$.

Theorem 3.3.3. (Vitali's covering theorem)

(a) Let F be an $\mathcal{H}^s$-measurable subset of $\mathbb{R}^d$ and let $\mathcal{V}$ be a Vitali class of closed sets for F. Then we may select a (finite or countable) disjoint sequence U_i from $\mathcal{V}$ such that either

$$\sum |U_i|^s = \infty \quad \textit{or} \quad \mathcal{H}^s\left(E \setminus \bigcup_i U_i\right) = 0.$$

(b) If $\mathcal{H}^s(F) < \infty$, then, given $\epsilon > 0$, we may also require that

$$\mathcal{H}^s(F) \leq \sum_i |U_i|^s + \epsilon.$$

Proof. Vitali's covering theorem is a well known result and its proof may be found in [Fal86]. □

3.3.2 Key Results for Local Convex Density

Theorem 3.3.5 below, the convex density analog to Theorem 3.2.6, becomes particularly important in Section 3.3.4 where we use it to prove Theorem 3.3.11.

Theorem 3.3.4. *If F is an s-set in $\mathbb{R}^n$, then $\overline{D}_c^s(F, x) = 0$ for $\mathcal{H}^s$-almost all $x \notin F$.*

Proof. The proof of this is omitted, but may be found in [Fal86]. □

Theorem 3.3.5. *If F is an s-set in $\mathbb{R}^d$, then $\overline{D}_c^s(F, x) = 1$ at $\mathcal{H}^s$-almost all $x \in F$.*

Proof.

"$\geq$" Fix $\alpha < 1$ and $p > 0$. Let

$$E = \left\{ x \in F : \frac{\mathcal{H}^s(F \cap U)}{|U|^s} < \alpha \text{ for all convex } U \text{ with } x \in U \text{ and } |U| \leq p \right\}$$

For any $\epsilon > 0$ we may find a p-cover of E by convex sets U_i such that $\sum_i |U_i|^s < \mathcal{H}^s(E) + \epsilon$. Hence, assuming each U_i contains some point of E,

$$\begin{aligned} \mathcal{H}^s(E) &\leq \sum_i \mathcal{H}^s(E \cap U_i) \\ &\leq \sum_i \mathcal{H}^s(F \cap U_i) \\ &< \alpha \sum_i |U_i|^s \\ &< \alpha(\mathcal{H}^s(E) + \epsilon) \end{aligned}$$

Since $\alpha < 1$ and this holds for all $\epsilon > 0$, $\mathcal{H}^s(E) = 0$. We can choose E for any

$p > 0$, so

$$\overline{D}_c^s(F, x) \geq \frac{\mathcal{H}^s(F \cap U)}{|U|^s} \geq \alpha$$

for $\mathcal{H}^s$-almost all $x \in F$. This is true for all $\alpha < 1$, so $\overline{D}_c^s(F, x) \geq 1$ for $\mathcal{H}^s$-almost all $x \in F$.

"$\leq$" This inequality is a bit more difficult and requires Vitali's covering theorem (Theorem 3.3.3).

We want to prove that $\overline{D}_c^s(F, x) \leq 1$ almost everywhere.

Let $E_1 = \{x \in F : \overline{D}_c^s(F, x) > 1\}$. If we can show that this set has Hausdorff measure zero, then we will have shown that $\overline{D}_c^s(F, x) \leq 1$ holds for $\mathcal{H}^s$-almost all $x \in F$. To do this, first we let $\alpha > 1$ be given and define another set as follows: $E_\alpha = \{x \in F : \overline{D}_c^s(F, x) > \alpha\}$. It is sufficient to show that $\mathcal{H}^s(E_\alpha) = 0$ for all $\alpha > 1$. To see why this is so we let

$$E_{1+\frac{1}{n}} = \left\{ x \in F : \overline{D}_c^s(F, x) > 1 + \frac{1}{n} \right\}.$$

Since $\overline{D}_c^s(F, x) > 1 + \frac{1}{n} > 1 + \frac{1}{n+1}$, $E_{1+\frac{1}{n}}$ sits inside $E_{1+\frac{1}{n+1}}$. Therefore we have an increasing sequence of sets as follows: $E_{1+\frac{1}{n}} \subseteq E_{1+\frac{1}{n+1}} \subseteq \cdots \subseteq E_1$. Clearly the union of all these sets is equal to E_1, so we have

$$\mathcal{H}^s(E_1) \leq \sum_{n=1}^{\infty} \mathcal{H}^s(E_{1+\frac{1}{n}}).$$

If $\mathcal{H}^s(E_{1+\frac{1}{n}}) = 0$ for all n, then $\mathcal{H}^s(E_1) = 0$. Thus if we can show that $\mathcal{H}^s(E_\alpha) = 0$ for all $\alpha > 1$, we will have shown that $\mathcal{H}^s(E_1) = 0$ as required.

Let E_0 be a subset of E_α as follows: $E_0 = \left\{ x \in E_\alpha : \overline{D}_c^s(F \setminus E_\alpha, x) = 0 \right\}$.

According to Theorem 3.3.4, $\mathcal{H}^s\left(\left\{ x \in E_\alpha : \overline{D}_c^s(\mathbb{R}^d \setminus E_\alpha, x) \neq 0 \right\}\right) = 0$.

But since $F \setminus E_\alpha \subseteq \mathbb{R}^d \setminus E_\alpha$,

$$\begin{aligned} E_\alpha \setminus E_0 &= \left\{x \in E_\alpha : \overline{D}_c^s(F \setminus E_\alpha, x) \neq 0\right\} \\ &\subseteq \left\{x \in E_\alpha : \overline{D}_c^s(\mathbb{R}^d \setminus E_\alpha, x) \neq 0\right\}. \end{aligned}$$

This implies that

$$\begin{aligned} \mathcal{H}^s\left(E_\alpha \setminus E_0\right) &\leq \mathcal{H}^s\left(\left\{x \in E_\alpha : \overline{D}_c^s(\mathbb{R}^d \setminus E_\alpha, x) \neq 0\right\}\right) \\ &= 0. \end{aligned} \tag{3.3.1}$$

Let U be some convex set. Since $(E_\alpha \cap U) \cup ((F \setminus E_\alpha) \cap U) = F \cap U$, by the countable additivity property of the Hausdorff measure, we have

$$\begin{aligned} \frac{\mathcal{H}^s(F \cap U)}{|U|^s} &= \frac{\mathcal{H}^s(E_\alpha \cap U)}{|U|^s} + \frac{\mathcal{H}^s((F \setminus E_\alpha) \cap U)}{|U|^s} \\ &\leq \sup_V \frac{\mathcal{H}^s(E_\alpha \cap V)}{|V|^s} + \sup_W \frac{\mathcal{H}^s((F \setminus E_\alpha) \cap W)}{|W|^s} \end{aligned}$$

where the suprema are taken over all convex sets V, W. This holds for all such convex sets UE, so taking supremum over such sets, we get

$$\sup_U \frac{\mathcal{H}^s(F \cap U)}{|U|^s} \leq \sup_V \frac{\mathcal{H}^s(E_\alpha \cap V)}{|V|^s} + \sup_W \frac{\mathcal{H}^s((F \setminus E_\alpha) \cap W)}{|W|^s}.$$

Then if we restrict the diameter of the sets such that $0 < |U|, |V|, |W| < r$. and take upper limits as $r \to 0$, we get

$$\overline{D}_c^s(F, x) \leq \overline{D}_c^s(E_\alpha, x) + \overline{D}_c^s(F \setminus E_\alpha, x)$$

for all $x \in F$. Since $E_0 \subseteq F$, if we restrict our attention to the $x \in E_0$, the above equation also holds. But since $\overline{D}_c^s(F \setminus E_\alpha, x) = 0$ and $\overline{D}_c^s(F, x) > \alpha$ for all $x \in E_0$, we have $\overline{D}_c^s(E_\alpha, x) \geq \overline{D}_c^s(F, x) > \alpha$.

We define a family of sets $\mathcal{V}$ as

$$\mathcal{V} = \left\{ U : U \text{ is closed and convex and } \frac{\mathcal{H}^s(E_\alpha \cap U)}{|U|^s} > \alpha \right\}$$

Let $y \in E_0$ and $\delta > 0$. Then

$$\begin{aligned} \alpha &< \overline{D}_c^s(E_\alpha, y) \\ &= \overline{\lim_{r \to 0}} \left\{ \sup_U \frac{\mathcal{H}^s(E_\alpha \cap U)}{|U|^s} \;\middle|\; U \text{ is open convex with } 0 < |U| < r \text{ and } y \in U \right\}, \end{aligned}$$

which means that there must exist an $r < \delta$ such that

$$\sup \left\{ \sup_U \frac{\mathcal{H}^s(E_\alpha \cap U)}{|U|^s} \;\middle|\; U \text{ is open convex with } 0 < |U| < r \text{ and } y \in U \right\} > \alpha.$$

Therefore, for all $y \in E_0$ and $\delta > 0$, there exists some set V such that $y \in V$ and $0 < |V| < \delta$, whose closure $\overline{V}$ is a member of $\mathcal{V}$, making $\mathcal{V}$ a Vitali class for E_0.

Since $E_0 \subseteq E_\alpha \subseteq F$ and F is an s-set, by part (b) of Theorem 3.3.3 (Vitali's covering theorem) we may, given $\epsilon > 0$, find a disjoint sequence of sets $\{V_i\}_i$ in $\mathcal{V}$ with $\mathcal{H}^s(E_0) \leq \sum_i |V_i|^s + \epsilon$.

Equation (3.3.1) tells us that $\mathcal{H}^s(E_\alpha \setminus E_0) = 0$, so $\mathcal{H}^s(E_\alpha) = \mathcal{H}^s(E_0)$. Thus, using the definition of $\mathcal{V}$, we have

$$\begin{aligned} \mathcal{H}^s(E_\alpha) &= \mathcal{H}^s(E_0) \\ &\leq \sum_i |V_i|^s + \epsilon \end{aligned}$$

$$
\begin{aligned}
&< \frac{1}{\alpha}\sum_i \mathcal{H}^s(E_\alpha \cap V_i) + \epsilon \\
&= \frac{1}{\alpha}\mathcal{H}^s\left(\bigcup_i (E_\alpha \cap V_i)\right) \qquad (3.3.2)\\
&\leq \frac{1}{\alpha}\mathcal{H}^s(E_\alpha) + \epsilon. \qquad (3.3.3)
\end{aligned}
$$

We get Equation (3.3.2) using the countable additive property of the Hausdorff measure by noting that the V_i sets are disjoint. The union of the disjoint V_i sets intersected with E_α is clearly a subset of E_α, so Equation (3.3.3) holds by the second property of measure. This holds for any $\epsilon > 0$, so $\mathcal{H}^s(E_\alpha) = 0$ if $\alpha > 1$ as required.

□

3.3.3 A New Upper Convex Density Result for Self-Similar Sets

Here we give a proof for a new result which gives an upper bound for the upper convex density of a self-similar set over all points x. Zhou gave a proof for a version of this theorem which worked under certain conditions in his 1998 paper on the calculation of the Hausdorff measure of the Koch curve [Zho98]. The result is proved in a more general setting here. A number of lemmas are required and are provided after the proof. Once again, we use this result later on in the proof of Theorem 3.3.11 in Section 3.3.4.

Theorem 3.3.6. (new result) *If K is a self-similar set satisfying the open set condition with similarity mappings $\{S_1, \ldots S_n\}$ and associated contraction ratios $\{r_i, \ldots, r_n\}$, and $s = \dim_H(K)$, then*

$$\mathcal{H}^s(K \cap U) \leq |U|^s$$

for all Borel U*. This also implies that* $\overline{D}_c^s(K,x) \leq 1$ *for all* x*.*

Proof. We will prove by contradiction. Let us assume that there exists a Borel set U such that

$$\mathcal{H}^s(K \cap U) > |U|^s.$$

Choose $\eta > 0$ such that

$$(1-\eta)\,\mathcal{H}^s(K \cap U) > |U|^s.$$

Fix $\delta > 0$ and choose k such that $|S_{\mathbf{j}}(U)| < \delta$ for all $|\mathbf{j}| = k$. Let

$$A = \left(\bigcup_{|\mathbf{j}|=k} S_{\mathbf{j}}(U) \cap K \right) \text{ and } B = \left(K \setminus \bigcup_{|\mathbf{j}|=k} S_{\mathbf{j}}(U) \right).$$

Clearly $K \subseteq A \cup B$.

Let $\lambda = \frac{1}{2}\,\eta\,\mathcal{H}^s(A)$. Choose a δ-cover $(V_i)_i$ of B so that

$$\sum_i |V_i|^s \leq \mathcal{H}^s_\delta(B) + \lambda.$$

Then $(S_{\mathbf{j}})_{|\mathbf{j}|=k} \cup (V_i)_i$ forms a δ-cover of K. So, using the scaling property of Hausdorff measure (Proposition 1.3.4) and the definition of a similarity (Definition 2.2.6) in a similar way to Lemma 3.3.8, we derive the following:

$$\begin{aligned} \mathcal{H}^s_\delta(K) &\leq \sum_{|\mathbf{j}|=k} |S_{\mathbf{j}}(U)|^s + \sum_i |V_i|^s \\ &\leq \sum_{|\mathbf{j}|=k} r_{\mathbf{j}}^s\,|U|^s + \mathcal{H}^s_\delta(B) + \lambda \end{aligned}$$

$$
\begin{aligned}
&< \sum_{|\mathbf{j}|=k} r_{\mathbf{j}}^s \,(1-\eta)\, \mathcal{H}^s(K\cap U) + \mathcal{H}^s_\delta(B) + \lambda \\
&= (1-\eta) \sum_{|\mathbf{j}|=k} r_{\mathbf{j}}^s \,\mathcal{H}^s(K\cap U) + \mathcal{H}^s_\delta(B) + \lambda \\
&= (1-\eta) \sum_{|\mathbf{j}|=k} \mathcal{H}^s(S_{\mathbf{j}}(K\cap U)) + \mathcal{H}^s_\delta(B) + \lambda.
\end{aligned}
$$

Below we combine Lemma 3.3.9 and 1.1.13 to get Equation (3.3.4). We then employ Proposition 3.3.7 for Inequality (3.3.5) and Lemma 3.3.8 for Inequality (3.3.6), to achieve our result.

$$
\begin{aligned}
\mathcal{H}^s_\delta(K) &\le (1-\eta) \sum_{|\mathbf{j}|=k} \mathcal{H}^s(S_{\mathbf{j}}(K\cap U)) + \mathcal{H}^s_\delta(B) + \lambda \\
&= (1-\eta)\, \mathcal{H}^s\left(\bigcup_{|\mathbf{j}|=k} S_{\mathbf{j}}(K\cap U) \right) + \mathcal{H}^s_\delta(B) + \lambda && (3.3.4) \\
&\le (1-\eta)\, \mathcal{H}^s\left(\bigcup_{|\mathbf{j}|=k} S_{\mathbf{j}}(U)\cap K \right) + \mathcal{H}^s_\delta(B) + \lambda && (3.3.5) \\
&= (1-\eta)\, \mathcal{H}^s(A) + \mathcal{H}^s_\delta(B) + \lambda \\
&= \mathcal{H}^s(A) + \mathcal{H}^s_\delta(B) - \eta\, \mathcal{H}^s(A) + \frac{1}{2}\,\eta\, \mathcal{H}^s(A) \\
&= \mathcal{H}^s(A) + \mathcal{H}^s_\delta(B) - \frac{1}{2}\,\eta\, \mathcal{H}^s(A) \\
&\le \mathcal{H}^s(A) + \mathcal{H}^s_\delta(B) - \frac{1}{2}\,\eta\, \frac{|U|^s}{1-\eta}. && (3.3.6)
\end{aligned}
$$

Finally, letting $\delta \to 0$ gives

$$
\mathcal{H}^s(K) \le \mathcal{H}^s(K) - \frac{1}{2}\,\eta\, \frac{|U|^s}{1-\eta}
$$

which is a contradiction.

□

The next proposition and the three lemmas that follow are required for the proof of Theorem 3.3.6 above.

Proposition 3.3.7. *If K is a self-similar set satisfying the open set condition and described by similarity mappings $\{S_1, \ldots S_n\}$, then given some set U we have*

$$S_{\mathbf{j}}(K \cap U) \subseteq K \cap S_{\mathbf{j}}(U).$$

Proof. Because K is self-similar, $S_{\mathbf{j}}(K \cap U) \subseteq K$. Obviously $K \cap U \subseteq U$, so applying $S_{\mathbf{j}}$ to both sides, we have that $S_{\mathbf{j}}(K \cap U) \subseteq S_{\mathbf{j}}(U)$. Therefore $S_{\mathbf{j}}(K \cap U) \subseteq K \cap S_{\mathbf{j}}(U)$. □

Lemma 3.3.8. *Let K be a self-similar set satisfying the open set condition with similarity mappings $\{S_1, \ldots, S_n\}$ and associated ratios $\{r_i, \ldots, r_n\}$. Let U be a Borel set. Choose $\eta > 0$ such that $(1 - \eta)\,\mathcal{H}^s(K \cap U) > |U|^s$. Let $\delta > 0$ and choose k such that $|S_{\mathbf{j}}| < \delta$ for all $|\mathbf{j}| = k$. Then*

$$\mathcal{H}^s\left(\bigcup_{|\mathbf{j}|=k} S_{\mathbf{j}}(U \cap K)\right) \geq \frac{|U|^s}{1 - \eta}.$$

Proof. We have

$$\mathcal{H}^s\left(\bigcup_{|\mathbf{j}|=k} S_{\mathbf{j}}(U \cap K)\right) = \sum_{|\mathbf{j}|=k} \mathcal{H}^s(S_{\mathbf{j}}(U \cap K)).$$

Letting $A \subseteq \mathbb{R}^d$ and $\alpha A = \{\alpha\, x : x \in A\}$ for $\alpha > 0$, by the scaling property for the Hausdorff measure (Proposition 1.3.4) and the definition of a similarity (Definition 2.2.6), we have $\mathcal{H}^s(S_i(A)) = \mathcal{H}^s(r_i A) = r_i^s \mathcal{H}^s(A)$.

This works for any set A, so it will work for $S_{i_1}(A), S_{i_2}(A), S_{i_3}(A)$ and so on. So for some string **j**, applying this repeatedly and making use of the binomial theorem, we have

$$\sum_{|\mathbf{j}|=k} \mathcal{H}^s(S_{\mathbf{j}}(U \cap K)) \quad = \quad \sum_{|\mathbf{j}|=k} r_{\mathbf{j}}\, \mathcal{H}^s(U \cap K)$$

$$\begin{aligned}
&\geq \left(\sum_{|\mathbf{j}|=k} r_{\mathbf{j}}\right)\frac{|U|^s}{1-\eta} \\
&= \left(\sum_{i=1}^{n} r_i^s\right)^k \frac{|U|^s}{1-\eta} \\
&= \frac{|U|^s}{1-\eta}.
\end{aligned}$$

□

Lemma 3.3.9. *Let K be a self-similar set satisfying the open set condition with similarity mappings $\{S_1, \ldots, S_n\}$ and associated ratios $\{r_i, \ldots, r_n\}$. Let U be a Borel set. Then*

$$\mathcal{H}^s\left(S_{\mathbf{i}}(K \cap U) \cap S_{\mathbf{j}}(K \cap U)\right) = 0 \ \textit{ for } \mathbf{i} \neq \mathbf{j} \textit{ and } |\mathbf{i}| = |\mathbf{j}|.$$

Proof. First of all we note the following: clearly $K \cap U \subseteq K$ which implies that $S_{\mathbf{i}}(K \cap U) \subseteq S_{\mathbf{i}}(K)$ for any string **i**. By the second property of measure we know that $\mathcal{H}^s(S_{\mathbf{i}}(K \cap U)) \leq \mathcal{H}^s(S_{\mathbf{i}}(K))$, so it is sufficient to prove that

$$\mathcal{H}^s\left(S_{\mathbf{i}}(K) \cap S_{\mathbf{j}}(K)\right) = 0 \ \text{ for } \mathbf{i} \neq \mathbf{j} \text{ and } |\mathbf{i}| = |\mathbf{j}|,$$

since $\mathcal{H}^s\left(S_{\mathbf{i}}(K \cap U) \cap S_{\mathbf{j}}(K \cap U)\right) \leq \mathcal{H}^s\left(S_{\mathbf{i}}(K) \cap S_{\mathbf{j}}(K)\right)$.

Let $\mathbf{i} = i_1 \ldots i_n$ and $\mathbf{j} = j_1 \ldots j_n$ be two strings of length $|\mathbf{i}| = |\mathbf{j}| = n$ and let $i_1 \ldots i_{k-1} = j_1 \ldots j_{k-1}$ such that the kth term in each string is the first term where $i_k \neq j_k$. Then

$$\begin{aligned}
&\mathcal{H}^s\left(S_{\mathbf{i}}(K) \cap S_{\mathbf{j}}(K)\right) \\
&= \mathcal{H}^s\left(S_{i_1} \cdots S_{i_{k-1}} S_{i_k} S_{i_{k+1}} \cdots S_{i_n}(K) \cap S_{j_1} \cdots S_{j_{k-1}} S_{j_k} S_{j_{k+1}} \cdots S_{j_n}(K)\right) \\
&= \mathcal{H}^s\left(S_{i_1} \cdots S_{i_{k-1}}\left(S_{i_k} S_{i_{k+1}} \cdots S_{i_n}(K) \cap S_{j_k} S_{j_{k+1}} \cdots S_{j_n}(K)\right)\right) \\
&= r_{i_1}^s r_{i_2}^s \cdots r_{i_{k-1}}^s \mathcal{H}^s\left(S_{i_k} S_{i_{k+1}} \cdots S_{i_n}(K) \cap S_{j_k} S_{j_{k+1}} \cdots S_{j_n}(K)\right) && (3.3.7) \\
&\leq r_{i_1}^s r_{i_2}^s \cdots r_{i_{k-1}}^s \mathcal{H}^s(S_{i_k}(K) \cap S_{j_k}(K)) && (3.3.8)
\end{aligned}$$

$$= 0. \tag{3.3.9}$$

We get Equation (3.3.7) using the scaling property of Hausdorff measure (Proposition 1.3.4). Equation (3.3.8) is clear since $S_{i_{k+1}} \cdots S_{i_n}(K) \subseteq S_{i_k}(K)$ and $S_{j_{k+1}} \cdots S_{j_n}(K) \subseteq S_{j_k}(K)$. Finally, Equation (3.3.9) is due to a result by Hutchinson in [Hut81] which says that if the open set condition holds, $\mathcal{H}^s(S_i(K) \cap S_j(K)) = 0$ for $i \neq j$. □

3.3.4 Further Convex Density Theorems for Self-Similar Sets

In this section we prove three important results related to the convex density of self-similar sets. The most important of these, Theorem 3.3.13, forms a link between the Hausdorff measure and a density formulation which is based on a self-similar measure. Self-similar measures are the measure analogue of self-similar sets and hence are quite convenient to work with. Such a result brings us a step closer to using density results to help calculate the Hausdorff measure of certain sets. Firstly, we define the density formulation as follows:

Definition 3.3.10. Let μ be a measure on some set. We define an *upper convex s-dimensional density with respect to μ at a point x* as follows:

$$\overline{d}_c^s(\mu, x) = \overline{\lim_{r \to 0}} \left\{ \sup \frac{\mu(U)}{|U|^s} \right\}$$

where the supremum is taken over all open convex sets U where $0 < |U| < r$ and $x \in U$.

Theorem 3.3.13 says that if we have a self-similar set K which satisfies the open set condition and a self-similar measure λ on K, then

$$\mathcal{H}^s(K) = \frac{1}{\sup_x \overline{d}_c^s(\lambda, x)}.$$

In Section 3.4, we review a case in the literature, [AS99], where the supremum in the above equation has been estimated for Cantor-like sets.

Expanding this notion further, in [ZF00] Zhou *et al* prove that given an s-set $E \subseteq \mathbb{R}^d$, there exists a sequence $\{U_n\}_n$ of Borel sets in $\mathbb{R}^d$ such that

$$\frac{1}{\mathcal{H}^s(E)} \leq \overline{\lim_{n\to\infty}} \frac{\mathcal{H}^s(E \cap U_n)}{\mathcal{H}^s(E)\,|U_n|^s}.$$

In order for this limit process to be useful, one needs to find U_n such that $c_n\mathcal{H}^s(E \cap U_n) = \mathcal{H}^s(E)$, where c_n is a constant. As Zhou et al remark, though this result provides a way to calculate the upper bounds of the Hausdorff measure of self-similar sets satisfying the open set condition, in general it is difficult to construct such suitable U_n. Though we will not be discussing this particular result further, we will now proceed and explore the other results we have mentioned above in more detail.

Theorem 3.3.11. *If K is a self-similar set satisfying the open set condition and $s = \dim_H K$, then*

$$\sup_x \overline{D}_c^s(K, x) = 1.$$

Proof. We start with the upper bound:

"$\leq$" This follows directly from Theorem 3.3.6.

"$\geq$" We know from Theorem 3.3.5 that $\overline{D}_c^s(K, x) = 1$ for $\mathcal{H}^s$-almost all x. Therefore letting $A = \{x \in K : \overline{D}_c^s(K, x) \neq 1\}$, $\mathcal{H}^s(A) = 0$. Owing to a result by Hutchinson, we then have the following:

$$\mathcal{H}^s(K \setminus A) = \mathcal{H}^s(K) > 0.$$

Of course if $K \setminus A$ has non-zero $\mathcal{H}^s$-measure, then it is non-empty. Therefore, there must exist some $y \in K \setminus A$ such that

$$\sup_x \overline{D}_c^s(K,x) \geq \overline{D}_c^s(K,y) = 1.$$

□

Theorem 3.3.12. *If K is a self-similar set satisfying the open set condition and $s = \dim_H(K)$, then*

$$\sup_{\substack{U \text{ open, convex} \\ U\cap K\neq\emptyset}} \frac{\mathcal{H}^s(K\cap U)}{|U|^s} = 1.$$

Proof.

"$\leq$" This follows directly from Theorem 3.3.6.

"$\geq$" We know from the proof of Theorem 3.3.11 that there must exist some $y \in \{x \in K : \overline{D}_c^s(K,x) = 1\}$ such that

$$\begin{aligned}\overline{D}_c^s(K,y) &= \overline{\lim_{r\to 0}} \left\{ \sup_{\substack{U \text{ open, convex} \\ y\in U \\ 0<|U|<r}} \frac{\mathcal{H}^s(K\cap U)}{|U|^s} \right\} \\ &= 1.\end{aligned}$$

Therefore given $\epsilon > 0$, there must also exist some $r > 0$ so that

$$\begin{aligned}\sup_{\substack{U \text{ open, convex} \\ U\cap K\neq\emptyset}} \frac{\mathcal{H}^s(K\cap U)}{|U|^s} &\geq \\ \sup_{\substack{U \text{ open, convex} \\ y\in U \\ 0<|U|<r}} \frac{\mathcal{H}^s(K\cap U)}{|U|^s} &\geq 1-\epsilon.\end{aligned}$$

This is true for all $\epsilon > 0$, so we have

$$\sup_{\substack{U \text{ is open and convex} \\ U\cap K\neq\emptyset}} \frac{\mathcal{H}^s(K\cap U)}{|U|^s} \geq 1.$$

□

Theorem 3.3.13. *Let K be a self-similar set satisfying the open set condition, let $(S_1, \ldots, S_n)$ be its associated similarity mappings and let r_i be the Lipschitz ratio of S_i. Let λ be the self-similar measure satisfying*

$$\lambda(A) = \sum_{i=1}^{m} r_i^s \lambda(S_i^{-1}(A))$$

for any measurable set A. Then

$$\mathcal{H}^s(K) = \frac{1}{\sup_x \overline{d}_c^s(\lambda, x)}.$$

Proof. Note that by Hutchinson [Hut81]

$$\lambda(A) = \frac{\mathcal{H}^s(K\cap A)}{\mathcal{H}^s(K)},$$

hence using Theorem 3.3.11 we have the following inequality for all x:

$$\begin{aligned}\sup_x \overline{d}_c^s(\lambda, x) &= \sup_x \overline{\lim_{r\to 0}} \left\{ \sup_{\substack{U \text{ open, convex} \\ x\in U \\ 0<|U|<r}} \frac{\lambda(U)}{|U|^s} \right\} \\ &= \sup_x \overline{\lim_{r\to 0}} \left\{ \sup_{\substack{U \text{ open, convex} \\ x\in U \\ 0<|U|<r}} \frac{\mathcal{H}^s(K\cap U)}{\mathcal{H}^s(K)|U|^s} \right\}\end{aligned}$$

$$= \frac{1}{\mathcal{H}^s(K)} \sup_x \overline{\lim_{r\to 0}} \left\{ \sup_{\substack{U \text{ open, convex} \\ x\in U \\ 0<|U|<r}} \frac{\mathcal{H}^s(K\cap U)}{|U|^s} \right\}$$
$$= \frac{1}{\mathcal{H}^s(K)} \sup_x \overline{D}_c^s(K,x)$$
$$= \frac{1}{\mathcal{H}^s(K)}.$$

Hence

$$\mathcal{H}^s(K) = \frac{1}{\sup_x \overline{d}_c^s(\lambda, x)}.$$

□

3.4 Calculating Hausdorff Measure of Fractal Sets: A History of the Problem

3.4.1 Hausdorff Measure of Cantor-like Sets

In [Bae94] and [Bae98], Soo Baek analysed the Hausdorff dimension of certain generalised Cantor sets. Sandra Meinershagen subsequently worked on finding the Hausdorff measure of these same sets in [Mei02]. To describe the type of Cantor set discussed in [Bae94] we let $I_\emptyset = [0,1]$, then obtain the left subinterval $I_{\sigma,1}$ and the right subinterval $I_{\sigma,2}$ by deleting a middle open subinterval of I_σ inductively for each $\sigma \in \{1,2\}^n$, where $n = 0,1,2,\cdots$. The set $F = \bigcap_{n=0}^{\infty} \bigcup_{\sigma\in\{1,2\}^n} I_\sigma$ is called a *perturbed Cantor set* when the lengths of each interval at the n-th level of the construction may differ from level to level, but the $I_{\sigma,i}$ sets share the same length when $\sigma \in \{1,2\}^n$ and $i = 1,2$. In [Bae98], this construction is generalised so that the length of the $I_{\sigma,i}$ intervals, and consequently the contraction ratios used to generate those intervals, may vary arbitrarily. This type of construction is referred to as a *deranged Cantor set*.

Baek makes use of a notion of dimension particular to such Cantor sets when achieving the results in [Bae94]. Using the perturbed Cantor set construction described in the previous paragraph, $a_{n+1} = \frac{|I_{\sigma,1}|}{|I_\sigma|}$ is the contraction ratio used to get the $I_{\sigma,1}$ intervals at the n-th level when $\sigma \in \{1,2\}^n$ and $b_{n+1} = \frac{|I_{\sigma,2}|}{|I_\sigma|}$ is the contraction ratio for the $I_{\sigma,2}$ intervals. Given a perturbed Cantor set F, we let

$$h^s(F) = \varliminf_{n\to\infty} \prod_{k=1}^{n} (a_k^s + b_k^s)$$
$$q^s(F) = \varlimsup_{n\to\infty} \prod_{k=1}^{n} (a_k^s + b_k^s).$$

We then define the *lower and upper Cantor dimensions* of F to be

$$\dim_{\underline{C}} F = \sup\{s > 0 \mid h^s(F) = \infty\} \text{ and}$$
$$\dim_{\overline{C}} F = \sup\{s > 0 \mid q^s(F) = \infty\}$$

respectively. Baek proves that $\dim_{\underline{C}} F = \dim_H F$ for all perturbed Cantor sets F in [Bae94]. In [Mei02], Meinershagen shows that the Hausdorff measure of F is actually equal to the covering measure h^s on F at the critical dimension. [Bae98] sees Baek investigate the Hausdorff measure of a certain weakly convergent deranged Cantor set which satisfies a condition that all the sequences of the solutions of some power equations related to the contraction ratios in its construction converge to some number. Baek shows that this number is in fact the Hausdorff measure of the set.

Elizabeth Ayer and Robert Strichartz discuss the exact Hausdorff measure and intervals of maximum density for certain types of Cantor sets in their 1999 paper [AS99]. The type of Cantor sets they work with are the attractors (invariant sets) of IFSs made up of contractions on $[0, 1]$ of the form $S_j(x) = p_j x + b_j$ where $j = 1, \ldots, m$. They assume a slight variation of the open set condition for their calculations: there exists an open interval I such that $S_j(I) \subseteq I$ and the images $S_j(I)$ are disjoint. Given an IFS $(S_1, \ldots, S_m)$ with invariant set K of the type described, given a self-similar measure μ on K which satisfies $\mu = \sum_{j=1}^{m} p_j^\alpha \mu \circ S_j^{-1}$ and assuming the modified version of the open set condition, Ayer and Strichartz work with the variation on the result in Theorem 3.3.13:

$$\frac{1}{\mathcal{H}^\alpha(K)} = \sup \left\{ \frac{\mu(J)}{|J|^\alpha} \;\middle|\; J \subseteq [0, 1] \right\}. \tag{3.4.1}$$

They say that an IFS satisfies the *finiteness property* if the above supremum is attained for some interval $S_{j_1} S_{j_2} \ldots S_{j_n}([0, 1])$ for some n. They then go on to show that the finiteness property holds in many cases under certain conditions, in particular if $p_1 = p_m$ or more generally, if $\log p_1$ and $\log p_m$ are commensurable numbers, i.e. if $\frac{\log p_1}{\log p_m}$ is rational. When the finiteness property holds, they also provide an estimate of the size of n in $S_{j_1} S_{j_2} \ldots S_{j_n}([0, 1])$. When the finiteness property does not hold, they demonstrate how to obtain a sequence of intervals with lengths approaching zero whose densities approximate the supremum in Equation (3.4.1) from below.

A couple of the results provided in [AS99] had already been proven by Jacques Marion in [Mar86].

Further studies of the Hausdorff measure of Cantor sets exist, including Soon-Mo Jung's 1999 paper, [Jun99]. Using a combinatorial method, Jung estimates the Hausdorff measures of various self-similar sets, including uniform Cantor sets.

3.4.2 Hausdorff Measure of Non-Trivial Fractals

Numerous papers have been written on the calculation of Hausdorff measure for more complicated sets. One of the first of these papers, [Mar87] by Marion, gave an estimate of the upper bound of the Hausdorff measure of a *Sierpinski gasket*. To construct this Sierpinski gasket, we start with an equilateral triangle $\triangle ABC$ with sides of length 1 and call it S_0. Joining the midpoints of the sides of S_0, we remove the open inverted equilateral triangle that is formed and call the remaining set S_1. We join the midpoints of each of the three triangles in S_1 in a similar way, remove the three open inverted equilateral triangles that are formed as a result and call the remaining set S_2. Repeating this process, we obtain $S_0 \supset S_1 \supset S_2 \supset \cdots \supset S_n \supset \cdots$. The non-empty set $S = \bigcap_{n \geq 0} S_n$ is called the Sierpinski gasket. Marion estimated that $\mathcal{H}^s(S) \leq (\frac{1}{6})3^s \approx 0.90508$ when $s = \dim_H S$ and speculated that this was the actual Hausdorff measure of the gasket at the critical dimension. In his 1997 paper [Zho97b], a Chinese scientist named Zuoling Zhou, some of whose work we will be analysing in detail in the sequel, found a better upper bound for $\mathcal{H}^s(S)$ that disproved Marion's conjecture. In a subsequent paper, [Zho97a], Zhou improved this estimate further so that $\mathcal{H}^s(S) \leq (\frac{25}{22})(\frac{6}{7})^s \approx 0.8900$ and in [ZF00] Zhou and Feng improved the estimate still further until they arrived at $\mathcal{H}^s(S) \leq 0.83078799$. Following that, in 2002, Zhou *et al* [JZZ02] derived a lower bound for the Hausdorff measure of S, $\mathcal{H}^s(S) \geq 0.5$.

Work has also been done on calculating the Hausdorff measure of a *Koch curve* at the critical dimension. To construct a general Koch curve, we take a line segment in $\mathbb{R}^2$, divide it into 3 segments, then draw a triangle which uses the middle segment as a base. We then draw smaller triangles on each side of the remaining set in a similar way. Repeating this process infinitely many times, we derive a Koch curve, K. Marion [Mar87] conjectured that when K is constructed in a particular way, $s = \dim_H K$, $\mathcal{H}^s(K) =$

$2^{s-2} \approx 0.5995$. Further progress on Koch curves was made in [Zho98] and [ZZ01].

Later on in Chapter 4 we analyse and improve upon the work of Zhou and Wu [ZW99] on the calculation of the Hausdorff measure of a Sierpinski carpet in $\mathbb{R}^2$.

Chapter 4

The Hausdorff Measure of a Sierpinski Carpet in $\mathbb{R}^2$

4.1 Introduction

In this chapter, we develop a method for calculating the Hausdorff measure of a Sierpinski carpet based on a Zhou and Wu's calculation in [ZW99], but using an alternative geometrical technique. As shall be seen in the succeeding chapter, the method can be extended naturally to a three-dimensional setting where the Hausdorff measure of a Sierpinski sponge can be calculated. The Sierpinski carpet which we deal with here is the same set as the one described in [ZW99], for which the authors calculated a Hausdorff measure of $\sqrt{2}$. Using the alternative method of calculation we present here, we arrive at the same conclusion. We compute the Hausdorff measure of this set by calculating the Hausdorff measure of its one-dimensional projection onto a line and relating this to a mass distribution over the original set. While the skeleton of the proof remains the same as that of Zhou and Wu, we have reduced the number of lemmas and theorems required from six to three and have replaced much of the numerical machinery used to get one of the key results with a more intuitive geometrical concept.

It should be noted that although the set described here may not share the aesthetic qualities of our intuitive notion of a ‘carpet’ and there are certainly more worthy candidates, the construction of the set is consistent with the classical definition of a Sierpinski carpet, so we retain the title here for consistency. For the same reasons, the Sierpinski ‘sponge’ we deal with in the subsequent chapter retains its name, even though it is probably best described as a rather sparse sandbox.

4.2 Notation and Set-up

We proceed by describing the Sierpinski carpet whose Hausdorff measure we wish to calculate. The reader should refer to Figure 4.2.1 while reading the following, as it illustrates the structure and labelling of the first two levels of the construction of the carpet.

Take a unit square in $\mathbb{R}^2$ that shares a vertex with the origin and that has two of its sides lying on the positive x-axis and positive y-axis respectively. Label this square $C_\emptyset$. We may divide each side of $C_\emptyset$ into four identical segments of sidelength $\frac{1}{4}$ to obtain 4^2 non-intersecting squares of equal size in $C_\emptyset$. Removing those squares that do not share a vertex with $C_\emptyset$, we are left with four remaining squares which we label C_1, C_{2_1}, C_{2_2} and C_3. Specifically, C_1 is the square that has the origin as one of its vertices, C_3 is the square that has $(1,1)$ as one of its vertices and C_{2_1} and C_{2_2} are the two remaining squares, named arbitrarily. This is the first level of the construction of the Sierpinski carpet. It shall become clear later on why we are using the strange subscripts. For the second level of the construction, we subdivide each of the squares C_1, C_{2_1}, C_{2_2} and C_3 into 16 smaller

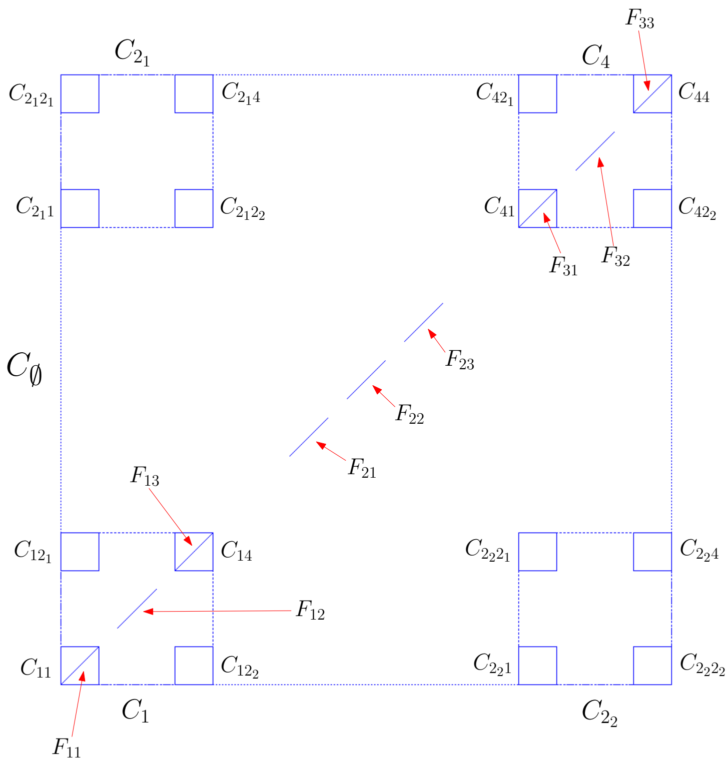


Figure 4.2.1: In this illustration, we show the unit square $C_\emptyset$ superimposed upon the first and second levels of the construction of the Sierpinski carpet C whose Hausdorff measure we are computing. The projection of the second level of the construction onto one of the main diagonals of the carpet is also shown. This projection is required for our calculations.

squares of sidelength $\frac{1}{4^2}$ or $\frac{1}{4}$ the sidelength of their parent squares, then remove those squares that do not share a vertex with the parent square C_i in each case. To illustrate the labelling of these remaining squares, we take the second level squares contained in an arbitrary square from the first level, C_1 say, and label them $C_{11}, C_{12_1}, C_{12_2}$ and C_{13} in the obvious way with C_{11} closest to the origin, C_{13} closest to $(1, 1)$ and of the remaining two squares, C_{12_1} closest to C_{2_1} and C_{12_2} closest to C_{2_2}. If we repeat this procedure infinitely many times we derive a Sierpinski carpet which we label C. It is clear that at the nth level of the construction we have 4^n squares of sidelength $\frac{1}{4^n}$ and we refer to these squares as the *basic squares* of the nth level. To refer to a specific basic square at the nth level, we use the notation $C_{j_1\cdots j_n}$. We use the following notation to refer to all of the basic squares at the nth level:

$$\mathcal{C}_n = \bigcup_{|\mathbf{j}|=n} C_{\mathbf{j}}.$$

Thus, clearly

$$C = \bigcap_{n=1}^{\infty} \mathcal{C}_n.$$

We select the diagonal between $(0,0)$ and $(1,1)$ to act as a main diagonal and label it $F_\emptyset$. This diagonal will become the focal point for much of the proof later on, specifically in Lemmas 4.5.2 and 4.5.3, but it should be noted that the calculations would work just as well for any of the other diagonals. We shall identify the main diagonal with the interval $[0,\sqrt{2}]$ in the obvious way for the purposes of our calculations. We must also define a mapping $\pi : C_\emptyset \to F_\emptyset$ to be the orthogonal projection from $\mathbb{R}^2$ onto $F_\emptyset$. Let the mappings $S_1, S_2, S_3 : F_\emptyset \to F_\emptyset$ be as follows:

$$\begin{aligned} S_1(x) &= \frac{1}{4}x \\ S_2(x) &= \frac{1}{4}x + \left(\frac{1}{4} + \frac{1}{8}\right)\sqrt{2} \\ S_3(x) &= \frac{1}{4}x + \frac{3}{4}\sqrt{2} \end{aligned}$$

Let $F_1 = S_1(F_\emptyset), F_2 = S_2(F_\emptyset)$ and $F_3 = S_3(F_\emptyset)$. We extend this notation so that $F_{i_1\cdots i_n} = S_{i_1\cdots i_n}(F_\emptyset)$, e.g. $F_{i_1 i_2} = S_{i_1 i_2}(F_\emptyset) = S_{i_1}(S_{i_2}(F_\emptyset))$. It is easily seen that at the first level of the construction of C, C_1 maps to F_1, C_{2_1} maps to F_2, C_{2_2} maps to F_2 and

C_3 maps to F_3 under π-projection. E.g.

$$\pi(C_1) = \left[0, \frac{\sqrt{2}}{4}\right] = S_1\left(\left[0, \sqrt{2}\right]\right) = S_1(F_\emptyset) = F_1.$$

Extending this idea, it is clear by inspection that given a $C_{j_1\cdots j_n}$, $F_{i_1\cdots i_n}$ is equivalent to $\pi(C_{j_1\cdots j_n})$, the projection of $C_{j_1\cdots j_n}$ onto the main diagonal, where $i_k = 1$ if $j_k = 1$, $i_k = 2$ if $j_k = 2_1$ or $j_k = 2_2$ and $i_k = 3$ if $j_k = 3$. Given an $F_{i_1\cdots i_n}$, we use the following notation to refer to the collection of basic squares at the nth level that intersect $\pi^{-1}(F_{i_1\cdots i_n})$, the pre-image of $F_{i_1\cdots i_n}$:

$$\bigcup_{\substack{|\mathbf{j}|=n \\ C_{\mathbf{j}}\cap\pi^{-1}(F_{i_1\cdots i_n})\neq\emptyset}} C_{\mathbf{j}} = \bigcup \left\{ C_{j_1\cdots j_n} \;\middle|\; \begin{array}{cc} j_k = 1 & \text{if } i_k = 1, \\ j_k = 2_1, 2_2 & \text{if } i_k = 2, \\ j_k = 3 & \text{if } i_k = 3. \end{array} \right\}$$

It can be shown that there exists a measure μ on C which acts as a mass distribution, distributing the mass $\sqrt{2}$ over C as follows:

$$\mu(C_{j_1\cdots j_n}) = \frac{1}{4^n}\sqrt{2}.$$

We have clearly defined μ for all of the $C_{j_1\cdots j_n}$ which are Borel subsets of $\mathbb{R}^2$, however, proving that μ is indeed a measure on *all* Borel sets in $\mathbb{R}^2$ is a much more involved task, one which we shall not be tackling here. Similarly, it can be shown that there exists a measure m supported on F such that

$$m(F_{i_1\cdots i_{n+1}}) = p_{i_1}\cdots p_{i_n}\sqrt{2}$$

where $p_{i_k} = \frac{1}{4}$ when $i_k = 1, 3$ and $p_{i_k} = \frac{1}{2}$ when $i_k = 2$. In particular,

$$m(F_{i_1 \cdots i_{n+1}}) = \frac{1}{4} m(F_{i_1 \cdots i_n}) \text{ when } i_{n+1} = 1, 3$$
$$m(F_{i_1 \cdots i_{n+1}}) = \frac{1}{2} m(F_{i_1 \cdots i_n}) \text{ when } i_{n+1} = 2.$$

We illustrate how the measure m may be constructed by dividing a mass of $\sqrt{2}$ over F_1, F_2 and F_3 so that they get mass $\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{2}$ and $\frac{\sqrt{2}}{4}$ respectively. Then the mass of each of the F_{i_1} is divided in a similar way amongst the $F_{i_1 i_2}$ such that, given an F_{i_1}, $F_{i_1 1}$ and $F_{1_1 3}$ get $\frac{1}{4}$ of its mass and $F_{i_1 2}$ gets $\frac{1}{2}$ of its mass. We repeat this procedure for each $F_{i_1 \cdots i_n}$ and its given mass, so that each $F_{i_1 \cdots i_{n+1}}$ receives either $\frac{1}{4}$ or $\frac{1}{2}$ of that mass as appropriate.

Let v be a vertex of $C_\emptyset$. A triangle is formed when we intersect $C_\emptyset$ with a line which is a distance x from v along the diagonal of C that runs through v and perpendicular to that diagonal. We refer to this triangle as $\triangle x$.

4.3 Main Result

Theorem 4.3.1. $\mathcal{H}^1(C) = \sqrt{2}$.

This result shall be proved in Section 4.6, but we require a number of other results first.

4.4 The Hausdorff Dimension of the Carpet

Proposition 4.4.1. $\dim_H C = 1$.

Proof. C is clearly a self-similar set under the four similarities $\{R_1, R_2, R_3, R_4\}$ with contraction ratios $\frac{1}{4}$ which map $C_\emptyset$ onto the four basic squares of the first level of the construction. Taking $C_\emptyset^\circ$ as the interior of $C_\emptyset$, the open set condition holds since

$$\bigcup_{i=1}^{4} R_i(C_\emptyset^\circ) \subset C_\emptyset^\circ$$

Then by Theorem 2.4.3,

$$s = \dim_H C = \dim_B C = 1,$$

the solution of $\sum_1^4 (\frac{1}{4})^s = 1$. □

4.5 Supportive Theorem and Lemmas

Theorem 4.5.1. *Let K be a self-similar set constructed using an IFS with similarity mappings $(R_1, \ldots, R_n)$ and associated contraction ratios $(c_1, \ldots, c_n)$. Let s be the unique real number such that*

$$\sum_{i=1}^{n} c_i^s = 1.$$

Then

$$\mathcal{H}^s(K) \leq \textit{diam}(K)^s.$$

In particular, $\textit{dim}_{\mathcal{H}}(K) \leq s$.

Proof. Let $K_{i_1 \cdots i_n} = R_{i_1 \cdots i_n}(K)$ and let $r_{i_1 \cdots i_n} = c_{i_1} c_{i_2} \cdots c_{i_n}$. Note that

$$\text{diam}(K_{i_1 \cdots i_n}) = \text{diam}(R_{i_1 \cdots i_n}(K)) \leq c_{i_1} \cdots c_{i_n} \text{diam}(K) = r_{i_1 \cdots i_n} \text{diam}(K).$$

We know that

$$K = \bigcup_{i=1}^{n} K_i = \bigcup_{i=1}^{n} R_i(K)$$

and applying this repeatedly we have

$$K = \bigcup_{|\mathbf{i}|=n} K_{\mathbf{i}}.$$

Clearly this union provides a cover of K. Fixing $\delta > 0$, we may then choose n large enough such that

$$\begin{aligned} \text{diam}(K_{i_1\cdots i_n}) &\leq r_{i_1\cdots i_n}\text{diam}(K) \\ &\leq c_{\max}^n \text{diam}(K) \\ &\leq \delta. \end{aligned}$$

$\bigcup_{|\mathbf{i}|=n} K_{\mathbf{i}}$ forms a δ-cover of K, hence

$$\begin{aligned} \mathcal{H}_\delta^s(K) &\leq \sum_{|\mathbf{i}|=n} \text{diam}(K_{\mathbf{i}})^s \\ &= \sum_{|\mathbf{i}|=n} r_{\mathbf{i}}\text{diam}(K)^s \\ &= (\sum_{i_1} c_{i_1}^s)\cdots(\sum_{i_n} c_{i_n}^s)\text{diam}(K)^s \\ &= (1)(1)\cdots(1)\text{diam}(K)^s \\ &= \text{diam}(K)^s. \end{aligned}$$

Hence $\mathcal{H}^s(K) \leq \text{diam}(K)^s$. □

Lemma 4.5.2. *$m(B) = \mu(\pi^{-1}(B))$ for all $B \subseteq F_\emptyset$, B are Borel sets.*

Proof. We prove this using the Carathéodory Uniqueness Theorem (Theorem 1.1.15). Firstly, let

$$\mathcal{I}_n = \{F_{i_1 \cdots i_n} \mid n \in \mathbb{N},\ i_1, \ldots, i_n = 1, \ldots, 3\}$$

and let

$$\mathcal{A}_n = \{I_1 \cup \cdots \cup I_m \mid m \in \mathbb{N},\ I_i \in \mathcal{I}_n\}.$$

Also let $\mathcal{I} = \bigcup_n \mathcal{I}_n$ and $\mathcal{A} = \left\{ \bigcup_{i=1}^{n} I_i \;\middle|\; I_i \in \mathcal{I} \right\}$.

Firstly we show that

$$m(A) = \mu(\pi^{-1}(A)) \text{ for all } A \in \mathcal{A}_n \tag{4.5.1}$$

To prove this , it is sufficient to show that

$$m(I) = \mu(\pi^{-1}(I)) \text{ for } I \in \mathcal{I}_n. \tag{4.5.2}$$

This is true since both m and $\mu \circ \pi^{-1}$ are measures with the countable additivity property. For example, if $A = I_1 \cup I_2 \cup I_3$, where $I_i \in \mathcal{I}_n$, then $m(A) = \sum_1^3 m(I_i)$ and $\mu(\pi^{-1}(A)) = \sum_1^3 \mu(\pi^{-1}(I_i))$ by countable additivity. To prove the result, we use an inductive process. First, we can easily see that statement (4.5.2) is true when $n = 0$:

$$\begin{aligned} m(F_\emptyset) &= \sqrt{2} \\ &= \mu(\pi^{-1}(F_\emptyset) \cap C_\emptyset) \\ &= \mu(\pi^{-1}(F_\emptyset)) \ldots \text{because } C_\emptyset \text{ has the only mass that lies in } \pi^{-1}(F_\emptyset). \end{aligned}$$

Next we assume that the statement is true for some $n \geq 0$ and prove it for $n + 1$. Thus we want to show that $m(F_{i_1\cdots i_{n+1}}) = \mu(\pi^{-1}(F_{i_1\cdots i_{n+1}}))$ for all $n \in \mathbb{N}$ where $i_k = 1, 2, 3$. This naturally breaks down into two distinct cases where either $i_{n+1} = 1, 3$ or $i_{n+1} = 2$.

Case 1: $i_{n+1} = 1, 3$

When $i_{n+1} = 1$ or $i_{n+1} = 3$ we have the following:

$$\mu(\pi^{-1}(F_{i_1\cdots i_{n+1}})) = \mu\left(\bigcup_{\substack{|\mathbf{j}|=n+1 \\ C_\mathbf{j}\cap\pi^{-1}(F_{i_1\cdots i_{n+1}})\neq\emptyset}} C_\mathbf{j}\right) \quad (4.5.3)$$

$$= \sum_{\substack{|\mathbf{j}|=n+1 \\ C_\mathbf{j}\cap\pi^{-1}(F_{i_1\cdots i_{n+1}})\neq\emptyset}} \mu(C_\mathbf{j}) \quad (4.5.4)$$

$$= \frac{1}{4} \sum_{\substack{|\mathbf{j}|=n \\ C_\mathbf{j}\cap\pi^{-1}(F_{i_1\cdots i_n})\neq\emptyset}} \mu(C_\mathbf{j}) \quad (4.5.5)$$

$$= \frac{1}{4}\,\mu\left(\bigcup_{\substack{|\mathbf{j}|=n \\ C_\mathbf{j}\cap\pi^{-1}(F_{i_1\cdots i_n})\neq\emptyset}} C_\mathbf{j}\right) \quad (4.5.6)$$

We get (4.5.3) by using the fact that the $C_{j_1\cdots j_{n+1}}$ squares that intersect the pre-image of $F_{i_1\cdots i_{n+1}}$ are the only objects with any μ-mass that lie the pre-image of $F_{i_1\cdots i_{n+1}}$. The countable additivity property of the μ measure allows us to sum the masses of the individual squares in (4.5.4). There is only one $C_{j_1\cdots j_{n+1}}$ square in each $C_{j_1\cdots j_n}$ square and it has $\frac{1}{4}$ of the mass of its parent square so we get (4.5.5). We use the countable additivity property of the μ-measure once again to derive (4.5.6). Next we look at the m-measure of $F_{i_1\cdots i_{n+1}}$:

$$m(F_{i_1\cdots i_{n+1}}) = \frac{1}{4}\, m(F_{i_i\cdots i_n}) \quad (4.5.7)$$

$$= \frac{1}{4}\,\mu(\pi^{-1}(F_{i_i\cdots i_n})) \tag{4.5.8}$$

$$= \frac{1}{4}\,\mu\left(\bigcup_{\substack{|\mathbf{j}|=n\\ C_{\mathbf{j}}\cap\pi^{-1}(F_{i_1\cdots i_n})\neq\emptyset}} C_{\mathbf{j}}\right) \tag{4.5.9}$$

$$= \mu(\pi^{-1}(F_{i_1\cdots i_{n+1}})) \tag{4.5.10}$$

When $i_{n+1} = 1, 4$, (4.5.7) follows by definition. (4.5.8) comes from our inductive assumption. Since the $C_{j_1\cdots j_n}$ that intersect the pre-image of $F_{i_1\cdots i_n}$ squares are the only objects that carry any mass in the pre-image of $F_{i_1\cdots i_n}$, we get (4.5.9). (4.5.10) follows directly from (4.5.6).

Case 2: $i_{n+1} = 2$

When $i_{n+1} = 2$, we have:

$$\mu(\pi^{-1}(F_{i_1\cdots i_{n+1}})) = \mu\left(\bigcup_{\substack{|\mathbf{j}|=n+1\\ C_{\mathbf{j}}\cap\pi^{-1}(F_{i_1\cdots i_{n+1}})\neq\emptyset}} C_{\mathbf{j}}\right) \tag{4.5.11}$$

$$= \sum_{\substack{|\mathbf{j}|=n+1\\ C_{\mathbf{j}}\cap\pi^{-1}(F_{i_1\cdots i_{n+1}})\neq\emptyset}} \mu(C_{\mathbf{j}}) \tag{4.5.12}$$

$$= \frac{2}{4} \sum_{\substack{|\mathbf{j}|=n\\ C_{\mathbf{j}}\cap\pi^{-1}(F_{i_1\cdots i_n})\neq\emptyset}} \mu(C_{\mathbf{j}}) \tag{4.5.13}$$

$$= \frac{1}{2}\,\mu\left(\bigcup_{\substack{|\mathbf{j}|=n\\ C_{\mathbf{j}}\cap\pi^{-1}(F_{i_1\cdots i_n})\neq\emptyset}} C_{\mathbf{j}}\right) \tag{4.5.14}$$

We get (4.5.11) because the $C_{j_1\cdots j_{n+1}}$ squares are the only objects with any μ-mass that lie the pre-image of $F_{i_1\cdots i_{n+1}}$. The countable additivity property of the μ measure allows us to sum the masses of the individual squares in (4.5.12). There are two $C_{j_1\cdots j_{n+1}}$ squares in each $C_{j_1\cdots j_n}$ square, each with equal mass which is $\frac{1}{4}$ of the mass of their parent square, so we get (4.5.13). Once again, countable additivity gets us (4.5.14). Looking at the m-measure of $F_{i_1\cdots i_{n+1}}$ when $i_{n+1} = 2$ we have:

$$
\begin{aligned}
m(F_{i_1\cdots i_{n+1}}) &= \frac{1}{2}\, m(F_{i_i\cdots i_n}) && (4.5.15)\\
&= \frac{1}{2}\, \mu(\pi^{-1}(F_{i_i\cdots i_n})) && (4.5.16)\\
&= \frac{1}{2}\, \mu\left(\bigcup_{\substack{|\mathbf{j}|=n \\ C_{\mathbf{j}} \cap \pi^{-1}(F_{i_1\cdots i_n}) \neq \emptyset}} C_{\mathbf{j}} \right) && (4.5.17)\\
&= \mu(\pi^{-1}(F_{i_1\cdots i_{n+1}})) && (4.5.18)
\end{aligned}
$$

When $i_{n+1} = 2$, (4.5.15) comes from our definition of m. (4.5.16) follows from the inductive assumption. Since the $C_{j_1\cdots j_n}$ squares are the only objects that carry any mass in the pre-image of $F_{i_1\cdots i_n}$, we get (4.5.17). (4.5.18) follows directly from (4.5.14).

This proves (4.5.1). Thus, we also have

$$m(A) = \mu(\pi^{-1}(A)) \text{ for all } A \in \mathcal{A}$$

which, according to Carathéodory's Uniqueness Theorem (Theorem 1.1.15), shows that

$$m(B) = \mu(\pi^{-1}(B)) \text{ for all } B \in \sigma(\mathcal{A}). \quad (4.5.19)$$

□

Lemma 4.5.3. $f(x) = m([0, x]) \geq \frac{4}{7}x$ *for all* $x \in [0, \sqrt{2}\,]$.

Proof. A graph of $f(x) = m([0, x])$ can be seen in Figure 4.5.1. We prove this result by dividing into four distinct cases. The last case is slightly more difficult to prove than the first three cases:

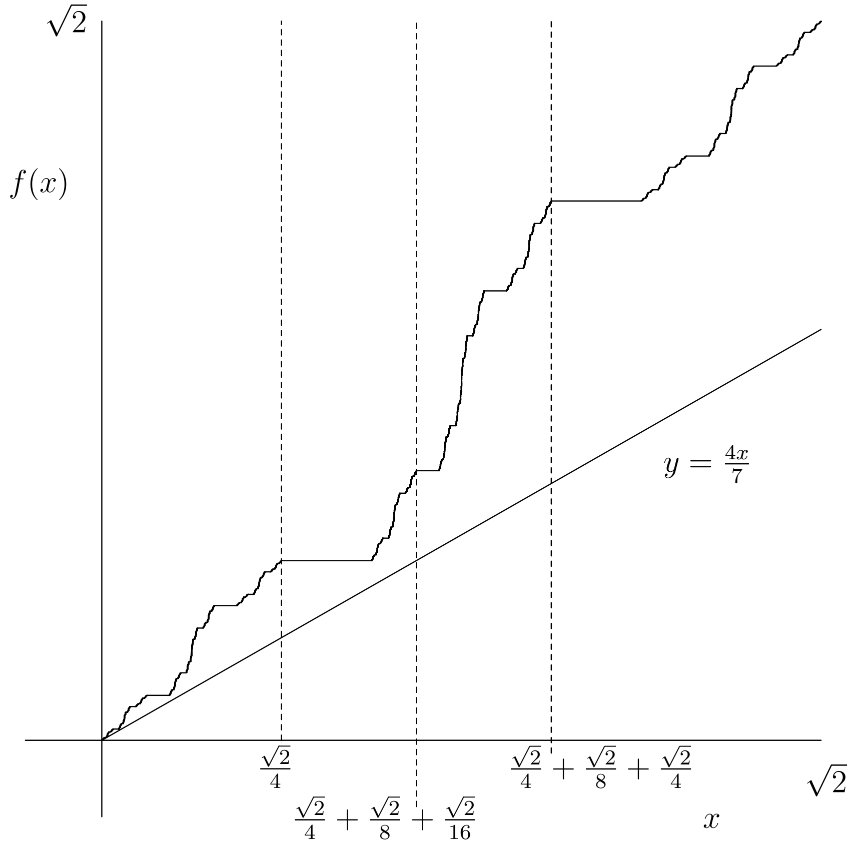


Figure 4.5.1: A graph of $f(x) = m([0, x])$ and the line $y = \frac{4}{7}x$ when $x \in [0, \sqrt{2}]$. The intervals used in each of the cases in the proof of Lemma 4.5.3 are also shown.

Case 1: $x \in \left[\frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{8} + \frac{\sqrt{2}}{4}, \sqrt{2}\right]$

We have:

$$\begin{aligned} f(x) &\geq f\left(\frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{8} + \frac{\sqrt{2}}{4}\right) \\ &= f(\frac{3\sqrt{2}}{4} - \frac{\sqrt{2}}{8}) \\ &= f(\frac{3\sqrt{2}}{4}) \end{aligned} \tag{4.5.20}$$

$$
\begin{aligned}
&= \frac{3\sqrt{2}}{4} \\
&\geq \frac{4}{7}x.
\end{aligned}
$$

We get (4.5.20) by noting that f is a monotonic increasing function.

Case 2: $x \in \left[\frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{8} + \frac{\sqrt{2}}{16}, \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{8} + \frac{\sqrt{2}}{4}\right]$

We have:

$$
\begin{aligned}
f(x) &\geq f\left(\frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{8} + \frac{\sqrt{2}}{16}\right) && (4.5.21) \\
&= \frac{7\sqrt{2}}{16} = \frac{\frac{7}{2}\sqrt{2}}{8} && (4.5.22) \\
&> \frac{4}{7}\left(\frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{8} + \frac{\sqrt{2}}{4}\right) = \frac{4}{7}\left(\frac{5\sqrt{2}}{8}\right) = \frac{\frac{20}{7}\sqrt{2}}{8} && (4.5.23) \\
&\geq \frac{4}{7}x.
\end{aligned}
$$

As in the previous case, we get (4.5.21) because f is monotonic increasing.

Case 3: $x \in \left[\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{8} + \frac{\sqrt{2}}{16}\right]$

We have:

$$
\begin{aligned}
f(x) &\geq f\left(\frac{\sqrt{2}}{4}\right) && (4.5.24) \\
&= \frac{\sqrt{2}}{4} \\
&= \sqrt{2}\left(\frac{4}{7}.\frac{7}{16}\right)
\end{aligned}
$$

$$\begin{aligned} &= \frac{4}{7}\left(\frac{\sqrt{2}}{4}+\frac{\sqrt{2}}{8}+\frac{\sqrt{2}}{16}\right) \\ &\geq \frac{4}{7}x. \end{aligned}$$

As in the previous two cases, we get (4.5.24) because f is monotonic increasing.

Case 4: $x \in \left[0, \frac{\sqrt{2}}{4}\right]$

Note that $\left[0, \frac{\sqrt{2}}{4}\right] = \bigcup_n S_1^n\left(\left[\frac{\sqrt{2}}{4}, \sqrt{2}\right]\right)$. We would like to show that

$$f(x) \geq \frac{1}{4}x \text{ when } x \in S_1^n\left(\left[\frac{\sqrt{2}}{4}, \sqrt{2}\right]\right) \text{ for all } n \geq 1.$$

We do this for $n = 1$, then prove by induction. First we show that f in the interval $[0, \frac{\sqrt{2}}{4}]$ is an S_1 re-scaling of f in the interval $[0, \sqrt{2}]$. Recall that $S_1(x) = \frac{1}{4}x$ and $S_1^{-1}(x) = 4x$ and note that $f(\frac{\sqrt{2}}{4}) = \frac{\sqrt{2}}{4}$. We want to show that $f(x) = S_1(f(4x))$ or that $f(x) = \frac{1}{4}f(4x)$ for all $x \in [0, \frac{\sqrt{2}}{4}]$. Letting $x \in [0, \frac{\sqrt{2}}{4}]$ we have:

$$\begin{aligned} f(x) &= m([0,x]) \\ &= \sum_{i=1}^{4} p_i\, m(S_i^{-1}([0,x])) \\ &= p_1\, m(S_1^{-1}([0,x])) \qquad (4.5.25) \\ &= \frac{1}{4}\, m([0,4x]) \\ &= \frac{1}{4}\, f(4x). \end{aligned}$$

Equation (4.5.25) comes as a direct result of S_1 being the only map that maps to the interval $[0, \frac{\sqrt{2}}{4}]$. Clearly the line $\frac{4}{7}x$ in the interval $[0, \sqrt{2}]$ rescales to $\frac{4}{7}x$ in $[0, \frac{\sqrt{2}}{4}]$ under

S_1 because for all $x \in [0, \frac{\sqrt{2}}{4}]$,

$$\begin{aligned} \frac{4}{7}x &= S_1(\tfrac{4}{7}.4x) \\ &= \frac{1}{4}.\frac{4}{7}.4x. \end{aligned}$$

So $f(x) \geq \frac{4}{7}x$ holds for all $x \in S_1([\frac{\sqrt{2}}{4}, \sqrt{2}]) = [\frac{\sqrt{2}}{16}, \frac{\sqrt{2}}{4}]$. We can now show that the inequality $f(x) \geq \frac{4}{7}x$ is also valid in the interval $[0, \frac{\sqrt{2}}{16}]$ by starting the induction. We assume that $f(x) \geq \frac{4}{7}x$ for all $x \in S_1^n([\frac{\sqrt{2}}{4}, \sqrt{2}])$ for some $n \geq 1$ and prove it for $n+1$. So we assume that

$$f(x) \geq \frac{4}{7}x \text{ for all } x \in S_1^n\left(\left[\tfrac{\sqrt{2}}{4}, \sqrt{2}\right]\right)$$

and aim to prove that

$$f(x) \geq \frac{4}{7}x \text{ for all } x \in S_1^{n+1}\left(\left[\tfrac{\sqrt{2}}{4}, \sqrt{2}\right]\right).$$

Let $x \in S_1^{n+1}([\frac{\sqrt{2}}{4}, \sqrt{2}])$. We have

$$\begin{aligned} f(x) &= m([0,x]) \\ &= \sum_{i=1}^{4} p_i\, m(S_i^{-1}([0,x])) \\ &= p_1\, m(S_1^{-1}([0,x])) \\ &= \frac{1}{4}\, m([0, S_1^{-1}(x)]) \\ &= \frac{1}{4}\, f(S_1^{-1}(x)). \end{aligned} \tag{4.5.26}$$

Since $S_1^{-1}(x) \in S_1^n([\frac{\sqrt{2}}{4}, \sqrt{2}])$ and our inductive assumption states that $f(x) \geq \frac{4}{7}x$ for all $x \in S_1^n([\frac{\sqrt{2}}{4}, \sqrt{2}])$, using (4.5.26) we conclude:

$$\begin{aligned} f(x) &= \frac{1}{4} f(S_1^{-1}(x)) \\ &\geq \frac{1}{4} \cdot \frac{4}{7} S_1^{-1}(x) \\ &= \frac{1}{4} \cdot \frac{4}{7} \cdot 4x \\ &= \frac{4}{7} x. \end{aligned}$$

We have shown that

$$\begin{aligned} f(x) \geq \tfrac{4}{7}x \text{ for all } x \in & \left[0, \tfrac{\sqrt{2}}{4}\right] \cup \\ & \left[\tfrac{\sqrt{2}}{4}, \tfrac{\sqrt{2}}{4} + \tfrac{\sqrt{2}}{8} + \tfrac{\sqrt{2}}{16}\right] \cup \\ & \left[\tfrac{\sqrt{2}}{4} + \tfrac{\sqrt{2}}{8} + \tfrac{\sqrt{2}}{16}, \tfrac{\sqrt{2}}{4} + \tfrac{\sqrt{2}}{8} + \tfrac{\sqrt{2}}{4}\right] \cup \\ & \left[\tfrac{\sqrt{2}}{4} + \tfrac{\sqrt{2}}{8} + \tfrac{\sqrt{2}}{4}, \sqrt{2}\right] \end{aligned}$$

thus completing the proof. □

Proposition 4.5.4. $\mu(\triangle x) \geq \frac{4}{7}x$

Proof. We can easily prove this using Lemma 4.5.2 and Lemma 4.5.3. Given a $\triangle x$,

$$\mu(\triangle x) = \mu(\pi^{-1}([0, x]))$$

where $[0, x] \subseteq F$. Using the two lemmas, we have

$$\mu(\pi^{-1}([0, x])) = m([0, x]) \geq \frac{4}{7}x$$

□

4.6 Proof of Main Result: Calculation of the Hausdorff Measure

Recall our main result, Theorem 4.3.1: $\mathcal{H}^1(C) = \sqrt{2}$. The proof is as follows:

Proof. We start the proof with the upper bound $\mathcal{H}^1(C) \leq \sqrt{2}$:

"$\leq$" This follows from Theorem 4.5.1.

"$\geq$" According to the mass distribution principle, if $\mu(V) \leq |V|$ for all measurable sets V, then $\mathcal{H}^1(C) \geq \mu(C)$.

Given a measurable set V, we proceed to show that $\mu(V) \leq |V|$ by dividing the problem into three distinct cases:

Case 1: V intersects exactly 2 C_j basic squares at the first level.

Case 1.1: The 2 C_j lie on one of the diagonals of $C_\emptyset$.

Let C_a and C_b be the two basic squares of $C_\emptyset$ lying on the diagonal. We have:

$$\mu(V) \leq \mu(C_a) + \mu(C_b) \tag{4.6.1}$$
$$= \frac{\sqrt{2}}{2} \tag{4.6.2}$$
$$= d(C_a, C_b) \tag{4.6.3}$$
$$\leq |V| \tag{4.6.4}$$

We get Equation (4.6.1) because C_a and C_b are the only basic squares that V intersects, thus the sum of their masses is the maximum mass V can attain. To get

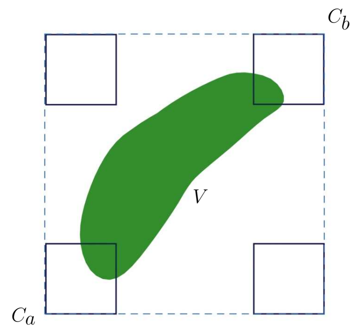


Figure 4.6.1: A set V intersects 2 C_j squares on one of the diagonals of $C_\emptyset$ at the first level of the construction of Sierpinski carpet C. This is Case 1.1 of the proof of Theorem 4.3.1.

Equation (4.6.2), we divide the total mass of C by two, since we have summed the masses of two of the four C_j basic squares at the first level. This is of course equivalent to the distance between C_a and C_b using the Hausdorff metric, since $d(C_a, C_b) = \sqrt{2} - 2(\frac{\sqrt{2}}{4})$, and so we have Equation (4.6.3). Finally, this distance is smaller than the diameter of V, since C_a and C_b intersect V.

Case 1.2: The two C_j basic squares lie on one of the sides of $C_\emptyset$.

Let C_a and C_b be the two basic squares which lie on that side of $C_\emptyset$. Without loss of generality, we assume that this side of $C_\emptyset$ is parallel to the x-axis. Let

$$\alpha = \inf \{x : (x, y) \in V \cap (C_a \cup C_b)\}$$

$$\beta = \sup \{x : (x, y) \in V \cap (C_a \cup C_b)\}$$

Let R_α denote the rectangle formed by the lines $x = \alpha$, $x = \frac{1}{4}$, $y = 0$ and $y = \frac{1}{4}$.
Let R_β denote the rectangle formed by the lines $x = \frac{3}{4}$, $x = \beta$, $y = 0$ and $y = \frac{1}{4}$.

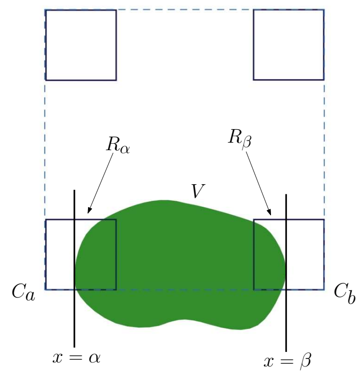


Figure 4.6.2: Case 1.2: V intersects 2 C_j squares on one of the sides of $C_\emptyset$ at the first level of the construction of C. Also shown are rectangles R_α and R_β.

When $0 \leq \alpha \leq \frac{1}{16}$, we have

$$\mu(R_\alpha) \leq \mu(C_a) = \frac{\sqrt{2}}{4} \tag{4.6.5}$$

since $R_\alpha \subseteq C_a$.

When $\frac{1}{16} \leq \alpha \leq \frac{1}{4}$, we have

$$\mu(R_\alpha) \leq \frac{\mu(C_a)}{2} = \frac{\sqrt{2}}{8} \tag{4.6.6}$$

since R_α can intersect at most two basic squares of C_a.

When $\frac{3}{4} \leq \beta \leq \frac{3}{4} + \frac{1}{16}$, we have

$$\mu(R_\beta) \leq \frac{\mu(C_b)}{2} = \frac{\sqrt{2}}{8} \tag{4.6.7}$$

since R_β can intersect at most two basic squares of C_b.

When $\frac{3}{4} + \frac{1}{16} \leq \beta \leq 1$ we have

$$\mu(R_\beta) \leq \mu(C_b) = \frac{\sqrt{2}}{4}. \tag{4.6.8}$$

Clearly $\mu(V) \leq \mu(R_\alpha) + \mu(R_\beta)$. Therefore, by the above results, we can show that $\mu(V) \leq |V|$ in the following cases:

When $0 \leq \alpha \leq \frac{1}{16}$, $\frac{3}{4} \leq \beta \leq \frac{3}{4} + \frac{1}{16}$, using Equations (4.6.5) and (4.6.7) to get Equation (4.6.9), we have

$$\begin{aligned} \mu \leq \mu(R_\alpha) + \mu(R_\beta) &\leq \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{8} = 2\frac{\sqrt{2}}{8} + \frac{\sqrt{2}}{8} \\ &= \frac{3\sqrt{2}}{8} = \frac{6\sqrt{2}}{16} \leq \frac{3}{4} - \frac{1}{16} \\ &\leq \beta - \alpha \\ &\leq |V|. \end{aligned} \tag{4.6.9}$$

When $0 \leq \alpha \leq \frac{1}{16}$, $\frac{1}{16} + \frac{3}{4} \leq \beta \leq 1$, using Equations (4.6.5) and (4.6.8) to get Equation (4.6.10), we have

$$\begin{aligned} \mu(V) \leq \mu(R_\alpha) + \mu(R_\beta) &\leq \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4} \\ &= \frac{8\sqrt{2}}{16} \leq \frac{1}{16} + \frac{3}{4} - \frac{1}{16} \\ &\leq \beta - \alpha \\ &\leq |V|. \end{aligned} \tag{4.6.10}$$

When $\frac{1}{16} \leq \alpha \leq \frac{1}{4}$, $\frac{3}{4} \leq \beta \leq \frac{3}{4} + \frac{1}{16}$, using Equations (4.6.6) and (4.6.7) to get Equation (4.6.11) we have

$$\mu(V) \leq \mu(R_\alpha) + \mu(R_\beta) \leq \frac{\sqrt{2}}{8} + \frac{\sqrt{2}}{8} \tag{4.6.11}$$

$$= \frac{\sqrt{2}}{4} \leq \frac{3}{4} - \frac{1}{4}$$
$$\leq \beta - \alpha$$
$$\leq |V|.$$

When $\frac{1}{16} \leq \alpha \leq \frac{1}{4}$, $\frac{1}{16} + \frac{3}{4} \leq \beta \leq 1$, using Equations (4.6.6) and (4.6.8) to get Equation (4.6.12) we have

$$\begin{aligned}\mu(V) &\leq \mu(R_\alpha) + \mu(R_\beta) \leq \frac{\sqrt{2}}{8} + \frac{\sqrt{2}}{4} \\ &= \frac{6\sqrt{2}}{16} \leq \frac{1}{16} + \frac{3}{4} - \frac{1}{4} \\ &\leq \beta - \alpha \\ &\leq |V|.\end{aligned} \tag{4.6.12}$$

This proves that $\mu(V) \leq |V|$ for all possible cases where V intersects C_a and C_b.

Case 2: V intersects 3 or 4 of the C_j basic squares at the first level.

Case 2.1: V intersects exactly 4 C_j basic squares at the first level.

We draw lines G_1, G_2, G_3 and G_4 through the vertex of each basic square of $C_\emptyset$ that lies in the interior of $C_\emptyset$, i.e. $C_\emptyset^\circ$. Without loss of generality, we will assume that G_1 is drawn through the inner vertex of of the basic square that also has a vertex at the origin. $G2, G_3$ and G_4 are drawn through the inner vertices of the basic squares that have vertices at $(1,0), (1,1)$ and $(0,1)$ respectively in a similar way. G_1 and G_3 should both be perpendicular to both G_2 and G_4.

We also draw lines A_1, A_2, A_3 and A_4 parallel respectively to G_1, G_2, G_3 and G_4, and obtain a rectangle that contains $\overline{V \cap C_\emptyset}$ and of which, each side intersects $\overline{V \cap C_\emptyset}$. This construction is illustrated in Figure 4.6.3.

Let

$$g_1 \quad = d(G_1, A_1), \quad a_1 = d((0,0), A_1),$$

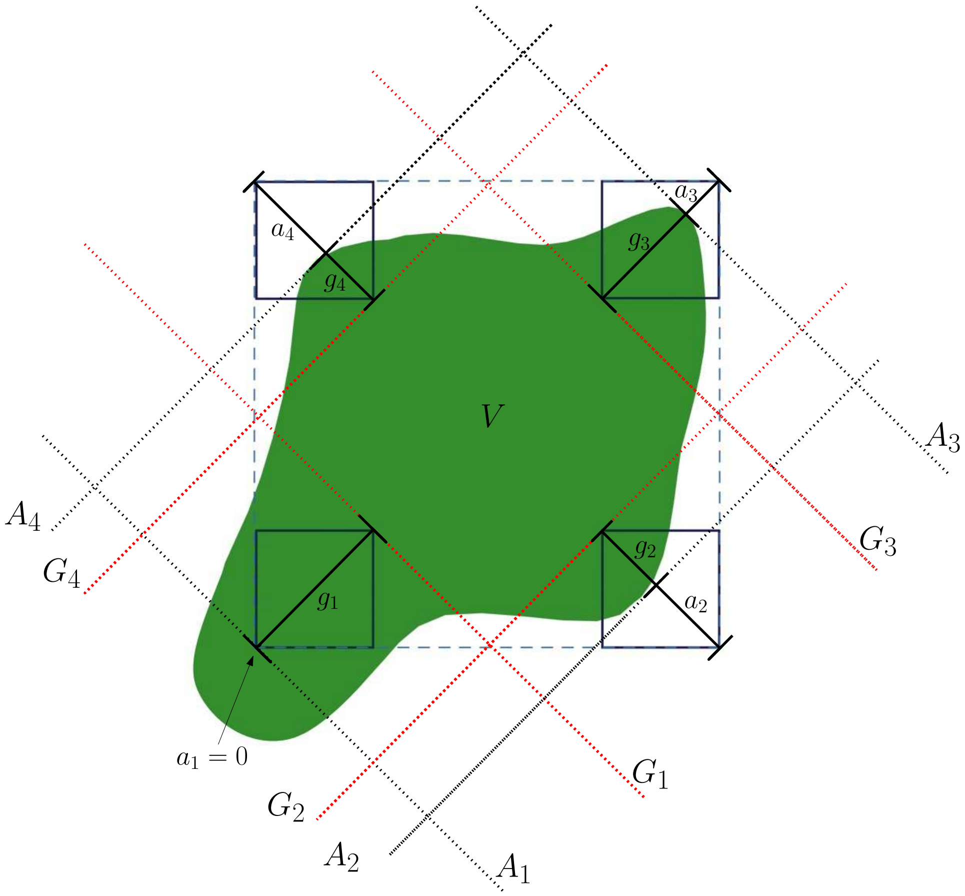


Figure 4.6.3: Case 2.1: V intersects 4 C_j squares at the first level of the construction of C. It is possible for some of V to lie outside of $C_\emptyset$ as is illustrated. The proof for this case requires the lines $A_1, A_2, A_3, A_4, G_1, G_2, G_3, G_4$ and the distances $a_1, a_2, a_3, a_4, g_1, g_2, g_3, g_4$.

$$\begin{aligned} g_2 &= d(G_2, A_2), \quad a_2 = d((1,0), A_2), \\ g_3 &= d(G_3, A_3), \quad a_3 = d((1,1), A_3), \\ g_4 &= d(G_4, A_4), \quad a_4 = d((0,1), A_4). \end{aligned}$$

We have

$$a_1 + g_1 = a_2 + g_2 = a_3 + g_3 = a_4 + g_4 = \frac{\sqrt{2}}{4}$$

and

$$|V| \geq \frac{\sqrt{2}}{2} + g_1 + g_3,$$
$$|V| \geq \frac{\sqrt{2}}{2} + g_2 + g_4.$$

Hence

$$\begin{aligned} 2|V| &\geq \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + g_1 + g_2 + g_3 + g_4 \\ |V| &\geq \frac{\sqrt{2}}{2} + \frac{1}{2}\left(g_1 + g_2 + g_3 + g_4\right) \\ &= \frac{\sqrt{2}}{2} + \frac{1}{2}\left(\sqrt{2} - (a_1 + a_2 + a_3 + a_4)\right) \qquad (4.6.13) \\ &= \sqrt{2} - \frac{1}{2}(a_1 + a_2 + a_3 + a_4). \qquad (4.6.14) \end{aligned}$$

We get Equation (4.6.13) by noting that

$$(g_1 + g_2 + g_3 + g_4) + (a_1 + a_2 + a_3 + a_4) = 4(\frac{\sqrt{2}}{4}) = \sqrt{2}.$$

By Proposition 4.5.4 and using Equation (4.6.14), we have

$$\begin{aligned} \mu(V) &\leq \sqrt{2} - (\mu(\triangle a_1) + \mu(\triangle a_2) + \mu(\triangle a_3) + \mu(\triangle a_4)) \\ &\leq \sqrt{2} - \frac{4}{7}\left(a_1 + a_2 + a_3 + a_4\right) \\ &\leq |V|. \end{aligned}$$

Case 2.2: V intersects exactly 3 C_j basic squares at the first level.

Without loss of generality we can assume that V intersects the three C_j basic squares that have vertices at $(0,0)$, $(0,1)$ and $(1,1)$ respectively. Similarly to Case 2.1, we draw lines G_1, G_2 and G_3 through the vertex of each of these basic squares that lies in the interior of $C_\emptyset^\circ$. Without loss of generality, we will assume that G_2 is

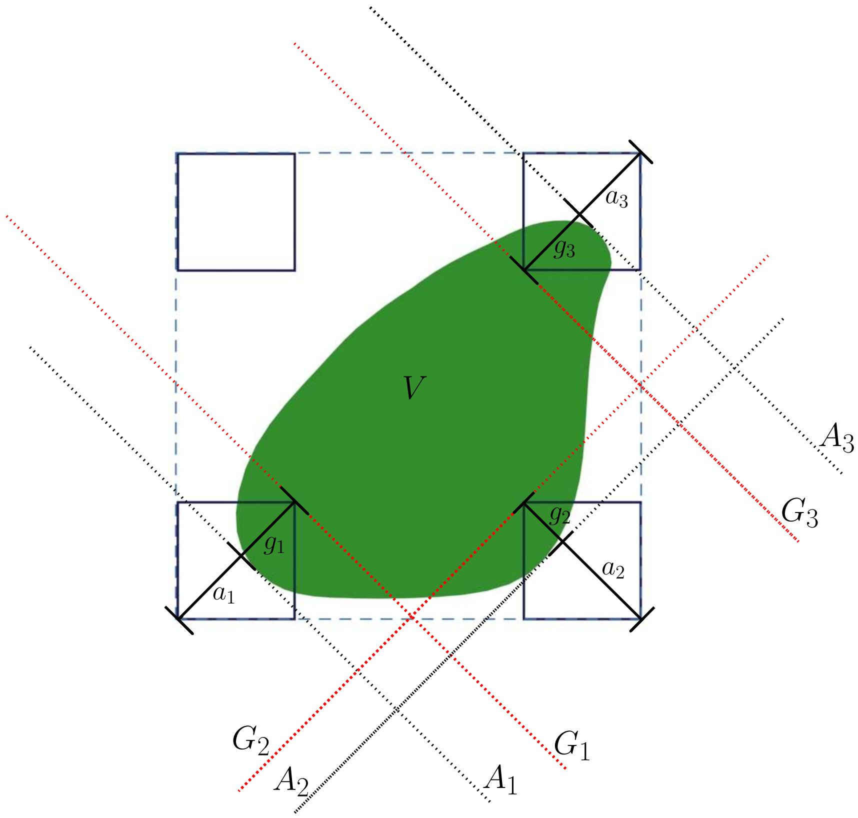


Figure 4.6.4: Case 2.2: V intersects 3 C_j squares at the first level of the construction of C. The proof for this case requires the lines $A_1, A_2, A_3, G_1, G_2, G_3$ and the distances $a_1, a_2, a_3, g_1, g_2, g_3$.

drawn through the inner vertex of the basic square that also has a vertex at $(1, 0)$, and is perpendicular to both G_1 and G_3.

We draw lines A_1, A_2 and A_3 parallel respectively to G_1, G_2 and G_3, and obtain a rectangle that contains $\overline{V \cap C_\emptyset}$ and of which, each side intersects $\overline{V \cap C_\emptyset}$. This is illustrated in Figure 4.6.4.

Letting

$$\begin{aligned} g_1 &= d(G_1, A_1), \quad a_1 = d((0,0), A_1), \\ g_2 &= d(G_2, A_2), \quad a_2 = d((1,0), A_2), \\ g_3 &= d(G_3, A_3), \quad a_3 = d((1,1), A_3), \end{aligned}$$

we get

$$a_1 + g_1 = a_2 + g_2 = a_3 + g_3 = \frac{\sqrt{2}}{4}$$

and

$$|V| \geq \frac{\sqrt{2}}{2} + g_1 + g_3 = \sqrt{2} - a_1 - a_3.$$

By Proposition 4.5.4, we have

$$\begin{aligned}
\mu(V) &\leq \frac{3}{4}\sqrt{2} - (\mu(\triangle a_1) + \mu(\triangle a_2) + \mu(\triangle a_3)) \\
&\leq \frac{3}{4}\sqrt{2} - \frac{4}{7}(a_1 + a_2 + a_3) \\
&\leq \frac{3}{4}\sqrt{2} - \frac{1}{2}(a_1 + a_2 + a_3),
\end{aligned}$$

so

$$\begin{aligned}
|V| - \mu(V) &\geq \sqrt{2} - a_1 - a_3 - \frac{3}{4}\sqrt{2} + \frac{1}{2}(a_1 + a_2 + a_3) \\
&= \frac{\sqrt{2}}{4} - \frac{1}{2}(a_1 + a_3) + \frac{1}{2}a_2 \\
&= \frac{1}{2}\left(\frac{\sqrt{2}}{2} - a_1 - a_3\right) + \frac{1}{2}a_2 \\
&\geq \frac{1}{2}a_2 \\
&\geq 0.
\end{aligned}$$

Case 3: V intersects exactly 1 C_j basic square at the first level.

This breaks down into 2 distinct subcases:

Case 3.1: V intersects 2, 3 or 4 $C_{j_1 j_2}$ basic squares at the second level.

Case 3.2: V intersects exactly 1 $C_{j_1 j_2}$ basic square at the second level.

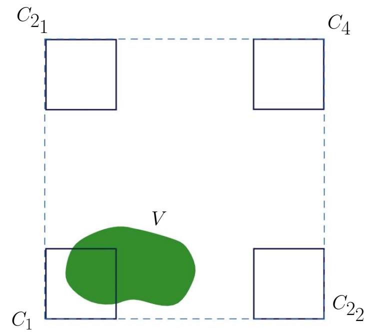


Figure 4.6.5: Case 3: V intersects 1 C_j square at the first level of the construction of C. To prove that $\mu(V) \leq |V|$ in this case requires that we look beyond the first level of the construction.

Proving Case 3.1 simply requires a repeat of the proofs in Case 1 and Case 2 over C_j instead of $C_\emptyset$. Proving Case 3.2 requires that we divide it into a further 2 subcases where V intersects either 2, 3 or 4 $C_{j_1 j_2 j_3}$ squares or exactly one $C_{j_1 j_2 j_3}$ square.

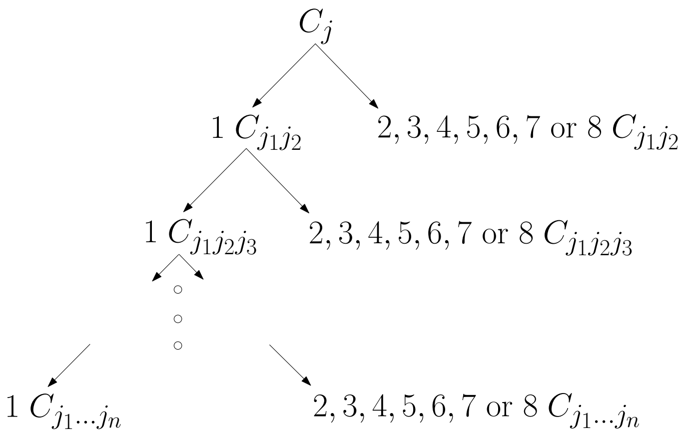


Figure 4.6.6: A tree representation of the proof of Case 3.

It is clear from Figure 4.6.6 that there are countably many pairs of subcases where either V intersects 2, 3 or 4 $C_{j_1 \dots j_n}$ or V intersects 1 $C_{j_1 \dots j_n}$. $\mu(V) \leq |V|$ when V intersects 2, 3 or 4 $C_{j_1 \cdots j_n}$ squares can be proven for all n by repeating the proofs for case 1 and case 2. When for all n, V intersects exactly one $C_{j_1 \cdots j_n}$ we have

$$V \subseteq \bigcup_n C_{j_1 \cdots j_n} = \{x\}$$

which is a singleton, and since the measure of a singleton is zero, $\mu(V) = 0 \leq |V|$ and we are done.

□

Chapter 5

The Hausdorff Measure of a Sierpinski Sponge in $\mathbb{R}^3$

This chapter forms the main body of the research component in this dissertation. The technique for calculating the Hausdorff measure of a two-dimensional Sierpinski carpet, outlined in the previous chapter, is extended for a similar calculation of the measure of a Sierpinski sponge, the three-dimensional analog of the carpet. We compute the Hausdorff measure of a sponge whose first level is constructed by using copies of the unit cube that have a sidelength of $\frac{1}{8}$th of the unit cube.

5.1 Notation and Set-Up

The Sierpinski sponge that we choose to compute the Hausdorff measure of here, is constructed as follows. Let $C_\emptyset$ be the closed unit cube in $\mathbb{R}^3$ that shares a vertex with the origin and that has three of its edges lying on the positive x-axis, the positive y-axis and the positive z-axis respectively. We divide $C_\emptyset$ into 8^3 cubes of equal size whose

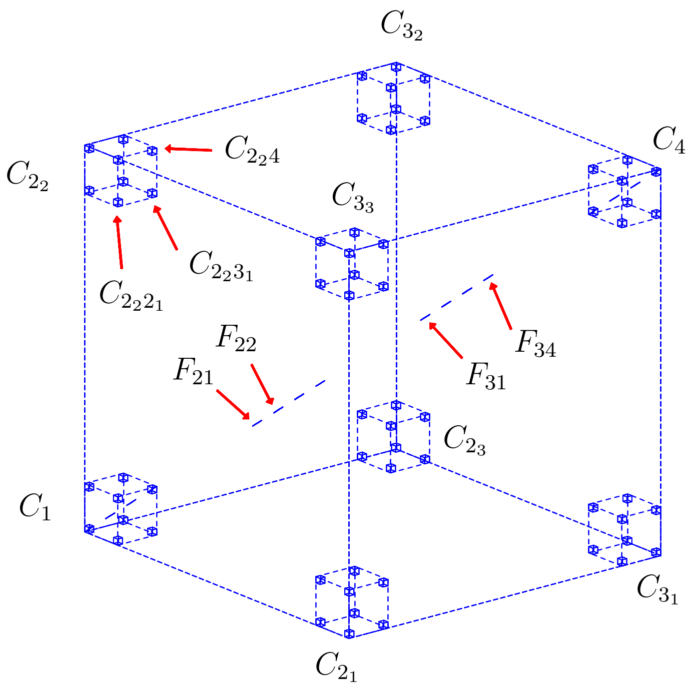


Figure 5.1.1: This figure shows unit cube $C_\emptyset$ superimposed on the first and second levels of the construction of the Sierpinski sponge whose Hausdorff measure we are computing. The projection of the second level of the construction onto one of the main diagonals of the sponge is also shown.

interiors do not intersect, each with sidelength $\frac{1}{8}$th of $C_\emptyset$, and remove those cubes that do not share a vertex with $C_\emptyset$, leaving 8 cubes remaining. To obtain the second level of the construction, we subdivide each of the 8 remaining cubes into 8^3 equally sized smaller cubes which do not intersect, with sidelength $\frac{1}{8}$th that of their parent cubes, and remove those cubes that do not share a vertex with their parent cubes, leaving 8^2 cubes remaining in total. Repeating this procedure at the second level of the construction and at each subsequent level of the construction yields a Sierpinski sponge which we label C. Clearly at the nth level of the construction there are 8^n cubes, each of sidelength 8^{-n}, which we shall refer to as the *basic cubes* of the nth level. We choose the diagonal of $C_\emptyset$ that shares a vertex with the origin to act as a main diagonal for our calculations and call it $F_\emptyset$. It should be noted that the results proved in Lemma 5.4.1 and Lemma 5.4.2 hold using any of the diagonals of $C_\emptyset$. For the purposes of our calculations, we shall identify the main diagonal with the interval $F_\emptyset = [0, \sqrt{3}]$ in the obvious way. Also, let $\pi : C_\emptyset \to F_\emptyset$ refer to the orthogonal projection from $\mathbb{R}^3$ onto $F_\emptyset$.

Define contraction mappings $S_1, S_2, S_3, S_4 : F_\emptyset \to F_\emptyset$ as follows:

$$\begin{aligned}
S_1(x) &= \frac{1}{8}x \\
S_2(x) &= \frac{1}{8}x + \left(\frac{1}{8} + \frac{1}{6}\right)\sqrt{3} \\
S_3(x) &= \frac{1}{8}x + \left(\frac{2}{8} + \frac{2}{6}\right)\sqrt{3} \\
S_4(x) &= \frac{1}{8}x + \left(\frac{2}{8} + \frac{3}{6}\right)\sqrt{3}
\end{aligned}$$

Let $F_1 = S_1(F_\emptyset), F_2 = S_2(F_\emptyset), F_3 = S_3(F_\emptyset)$ and $F_4 = S_4(F_\emptyset)$. At the first level of the construction, it is easy to see the basic cube that shares a vertex with the origin maps to F_1 under π-projection. We call this basic cube C_1. It is also easy to see that the three basic cubes nearest C_1 all map to F_2 under π-projection, so we call these cubes C_{2_1}, C_{2_2} and C_{2_3} respectively. The next three closest basic cubes to C_1 each map to F_3, so we label these C_{3_1}, C_{3_2} and C_{3_3}. The remaining basic cube at this first level of the construction maps to F_4 under π-projection, so we call it C_4.

Taking an arbitrary basic cube at the first level, C_j, we refer to the second level basic cubes contained therein as $C_{j1}, C_{j2_1}, C_{j2_2}, C_{j2_3}, C_{j3_1}, C_{j3_2}, C_{j3_3}$ and C_{j4} which are positioned in a similar way relative to the main diagonal as the first level basic cubes. Extending this notation, we may refer to any arbitrary basic cube at any level of the construction as $C_{j_1 \cdots j_n}$ where $j_k = 1, 2_1, 2_2, 2_3, 3_1, 3_2, 3_3, 4$. We label the union of all basic cubes at the nth level as follows:

$$\mathcal{C}_n = \bigcup_{|\mathbf{j}|=n} C_{\mathbf{j}}$$

and clearly

$$C = \bigcap_{n=1}^{\infty} \mathcal{C}_n.$$

Given a $C_{j_1\cdots j_n}$, $F_{i_1\cdots i_n}$ refers to $\pi(C_{j_1\cdots j_n})$, the projection of $C_{j_1\cdots j_n}$ onto the main diagonal, where $i_k = 1$ if $j_k = 1$, $i_k = 2$ if $j_k = 2_1, 2_2, 2_3$, $i_k = 3$ if $j_k = 3_1, 3_2, 3_3$, $i_k = 4$ if $j_k = 4$. Given an $F_{i_1\cdots i_n}$, we can see that the union of all basic cubes at the nth level which intersect $\pi^{-1}(F_{i_1\cdots i_n})$, the pre-image of $F_{i_1\cdots i_n}$, is given by:

$$\bigcup_{\substack{|\mathbf{j}|=n \\ C_{\mathbf{j}} \cap \pi^{-1}(F_{i_1\cdots i_n}) \neq \emptyset}} C_{\mathbf{j}} = \bigcup \left\{ C_{j_1\cdots j_n} \;\middle|\; \begin{array}{cl} j_k = 1 & \text{if } i_k = 1, \\ j_k = 2_1, 2_2, 2_3 & \text{if } i_k = 2, \\ j_k = 3_1, 3_2, 3_3 & \text{if } i_k = 3, \\ j_k = 4 & \text{if } i_k = 4. \end{array} \right\}$$

The construction and notation described above is illustrated in Figure 5.1.1. It is easily seen that there exists a measure μ supported on C such that

$$\mu(C_{j_1\cdots j_n}) = \frac{1}{8^n}\sqrt{3}.$$

Similarly, it is easily seen that there exists a measure m supported on F such that

$$m(F_{i_1\cdots i_{n+1}}) = p_{i_1} \cdots p_{i_n}\sqrt{3}$$

where $p_{i_k} = \frac{1}{8}$ when $i_k = 1, 4$ and $p_{i_k} = \frac{3}{8}$ when $i_k = 2, 3$. In particular,

$$m(F_{i_1\cdots i_{n+1}}) = \frac{1}{8} m(F_{i_1\cdots i_n}) \text{ when } i_{n+1} = 1, 4$$
$$m(F_{i_1\cdots i_{n+1}}) = \frac{3}{8} m(F_{i_1\cdots i_n}) \text{ when } i_{n+1} = 2, 3.$$

We can construct the m measure as follows. We divide a mass of $\sqrt{3}$ over F_1, F_2, F_3

and F_4, such that they get mass $\frac{\sqrt{3}}{8}$, $\frac{3\sqrt{3}}{8}$, $\frac{3\sqrt{3}}{8}$ and $\frac{\sqrt{3}}{8}$ respectively. The mass of each F_{i_1} is divided amongst the $F_{i_1 i_2}$ such that both $F_{i_1 1}$ and $F_{i_1 4}$ get $\frac{1}{8}$ of its mass, and $F_{i_1 2}$ and $F_{i_1 3}$ get $\frac{3}{8}$ of its mass. This process is repeated for each $F_{i_1 \cdots i_n}$ in a similar way, such that $F_{i_1 \cdots i_n 1}$, $F_{i_1 \cdots i_n 2}$, $F_{i_1 \cdots i_n 3}$ and $F_{i_1 \cdots i_n 4}$ receive masses of $\frac{1}{8}$, $\frac{3}{8}$, $\frac{3}{8}$ and $\frac{1}{8}$ of the mass of $F_{i_1 \cdots i_n}$ respectively.

Choose a vertex v of $C_\emptyset$. If we intersect $C_\emptyset$ with a plane perpendicular to the diagonal that passes through v, at a distance x from v, a pyramid is formed between v and the points of intersection. This pyramid is denoted by $\triangle x$.

5.2 Main Result

Theorem 5.2.1. $\mathcal{H}^1(C) = \sqrt{3}$.

This result is proved in Section 5.5 after a number of supportive lemmas are presented.

5.3 The Hausdorff Dimension of the Sponge

Proposition 5.3.1. $\dim_H C = 1$.

Proof. C is clearly a self-similar set under the eight similarities with contraction ratios $\frac{1}{8}$ which map $C_\emptyset$ onto the eight basic cubes of the first level of the construction. Using $C_\emptyset^\circ$, the interior of $C_\emptyset$, to fulfill the open set condition, by Theorem 2.4.3 we have

$$s = \dim_H C = \dim_B C = 1,$$

the solution of $\sum_1^8 (\frac{1}{8})^s = 1$. □

5.4 Supportive Lemmas

Lemma 5.4.1. $m(B) = \mu(\pi^{-1}(B))$ *for all* $B \subseteq F_\emptyset$, B *are Borel sets.*

Proof. Let $\mathcal{I}_n = \{F_{i_1 \cdots i_n} \mid n \in \mathbb{N},\ i_1, \ldots, i_n = 1, \ldots, 4\}$

and let $\mathcal{A}_n = \{I_1 \cup \cdots \cup I_m \mid m \in \mathbb{N},\ I_i \in \mathcal{I}_n\}$.

Also let

$$\mathcal{I} = \bigcup_n \mathcal{I}_n \quad \text{and} \quad \mathcal{A} = \left\{ \bigcup_{i=1}^{n} I_i \;\middle|\; I_i \in \mathcal{I} \right\}.$$

Firstly we show that

$$m(A) = \mu(\pi^{-1}(A)) \text{ for all } A \in \mathcal{A}_n \tag{5.4.1}$$

To prove this , it is sufficient to show that

$$m(I) = \mu(\pi^{-1}(I)) \text{ for } I \in \mathcal{I}_n$$

since both m and $\mu \circ \pi^{-1}$ are measures with the countable additivity property. To do this, we use an inductive process. First, we can easily see that the statement is true when $n = 0$:

$$\begin{aligned} m(F_\emptyset) &= \sqrt{3} \\ &= \mu(\pi^{-1}(F_\emptyset) \cap C_\emptyset) \\ &= \mu(\pi^{-1}(F_\emptyset)) \ldots \text{because } C_\emptyset \text{ has the only mass that lies in } \pi^{-1}(F_\emptyset). \end{aligned}$$

Next we assume that the statement is true for some $n \geq 0$ and prove it for $n+1$. Thus we want to show that $m(F_{i_1 \cdots i_{n+1}}) = \mu(\pi^{-1}(F_{i_1 \cdots i_{n+1}}))$ for all $n \in \mathbb{N}$ where $i_k = 1, 2, 3, 4$.

This naturally breaks down into two distinct cases where either $i_{n+1} = 1, 4$ or $i_{n+1} = 2, 3$.

Case 1: $i_{n+1} = 1, 4$

When $i_{n+1} = 1$ or $i_{n+1} = 4$ we have the following:

$$\mu(\pi^{-1}(F_{i_1\cdots i_{n+1}})) = \mu\left(\bigcup_{\substack{|\mathbf{j}|=n+1 \\ C_\mathbf{j}\cap\pi^{-1}(F_{i_1\cdots i_{n+1}})\neq\emptyset}} C_\mathbf{j}\right) \tag{5.4.2}$$

$$= \sum_{\substack{|\mathbf{j}|=n+1 \\ C_\mathbf{j}\cap\pi^{-1}(F_{i_1\cdots i_{n+1}})\neq\emptyset}} \mu(C_\mathbf{j}) \tag{5.4.3}$$

$$= \frac{1}{8} \sum_{\substack{|\mathbf{j}|=n \\ C_\mathbf{j}\cap\pi^{-1}(F_{i_1\cdots i_n})\neq\emptyset}} \mu(C_\mathbf{j}) \tag{5.4.4}$$

$$= \frac{1}{8}\,\mu\left(\bigcup_{\substack{|\mathbf{j}|=n \\ C_\mathbf{j}\cap\pi^{-1}(F_{i_1\cdots i_n})\neq\emptyset}} C_\mathbf{j}\right) \tag{5.4.5}$$

We get (5.4.2) by using the fact that the $C_{j_1\cdots j_{n+1}}$ cubes are the only objects with any μ-mass that lie the pre-image of $F_{i_1\cdots i_{n+1}}$. The countable additivity property of the μ measure allows us to sum the masses of the individual cubes in (5.4.3). There is only one $C_{j_1\cdots j_{n+1}}$ cube in each $C_{j_1\cdots j_n}$ cube and it has $\frac{1}{8}$ of the mass of its parent cube so we get (5.4.4). We use the countable additivity property of the μ-measure once again to derive (5.4.5). Next we look at the m-measure of $F_{i_1\cdots i_{n+1}}$:

$$m(F_{i_1\cdots i_{n+1}}) = \frac{1}{8}\, m(F_{i_i\cdots i_n}) \tag{5.4.6}$$

$$= \frac{1}{8}\, \mu(\pi^{-1}(F_{i_i\cdots i_n})) \tag{5.4.7}$$

$$= \frac{1}{8}\,\mu\left(\bigcup_{\substack{|\mathbf{j}|=n \\ C_{\mathbf{j}}\cap\pi^{-1}(F_{i_1\cdots i_n})\neq\emptyset}} C_{\mathbf{j}}\right) \quad (5.4.8)$$

$$= \mu(\pi^{-1}(F_{i_1\cdots i_{n+1}})) \quad (5.4.9)$$

When $i_{n+1} = 1, 4$, (5.4.6) follows by definition. (5.4.7) comes from our inductive assumption. Since the $C_{j_1\cdots j_n}$ cubes are the only objects that carry any mass in the pre-image of $F_{i_1\cdots i_n}$, we get (5.4.8). (5.4.9) follows directly from (5.4.5).

Case 2: $i_{n+1} = 2, 3$

When $i_{n+1} = 2$ or $i_{n+1} = 3$, we have:

$$\mu(\pi^{-1}(F_{i_1\cdots i_{n+1}})) = \mu\left(\bigcup_{\substack{|\mathbf{j}|=n+1 \\ C_{\mathbf{j}}\cap\pi^{-1}(F_{i_1\cdots i_{n+1}})\neq\emptyset}} C_{\mathbf{j}}\right) \quad (5.4.10)$$

$$= \sum_{\substack{|\mathbf{j}|=n+1 \\ C_{\mathbf{j}}\cap\pi^{-1}(F_{i_1\cdots i_{n+1}})\neq\emptyset}} \mu(C_{\mathbf{j}}) \quad (5.4.11)$$

$$= \frac{3}{8}\sum_{\substack{|\mathbf{j}|=n \\ C_{\mathbf{j}}\cap\pi^{-1}(F_{i_1\cdots i_n})\neq\emptyset}} \mu(C_{\mathbf{j}}) \quad (5.4.12)$$

$$= \frac{3}{8}\,\mu\left(\bigcup_{\substack{|\mathbf{j}|=n \\ C_{\mathbf{j}}\cap\pi^{-1}(F_{i_1\cdots i_n})\neq\emptyset}} C_{\mathbf{j}}\right) \quad (5.4.13)$$

We get (5.4.10) because the $C_{j_1\cdots j_{n+1}}$ cubes are the only objects with any μ-mass that lie the pre-image of $F_{i_1\cdots i_{n+1}}$. The countable additivity property of the μ measure allows us to sum the masses of the individual cubes in (5.4.11). There are three $C_{j_1\cdots j_{n+1}}$ cubes in each $C_{j_1\cdots j_n}$ cube, each with equal mass which is $\frac{1}{8}$ of the mass of their parent cube, so we get (5.4.12). Once again, countable additivity gets us (5.4.13). Looking at the m-measure of $F_{i_1\cdots i_{n+1}}$ when $i_{n+1} = 2, 3$ we have:

$$\begin{aligned} m(F_{i_1\cdots i_{n+1}}) &= \frac{3}{8}\, m(F_{i_i\cdots i_n}) && (5.4.14)\\ &= \frac{3}{8}\, \mu(\pi^{-1}(F_{i_i\cdots i_n})) && (5.4.15)\\ &= \frac{3}{8}\, \mu\left(\bigcup_{\substack{|\mathbf{j}|=n \\ C_{\mathbf{j}}\cap\pi^{-1}(F_{i_1\cdots i_n})\neq\emptyset}} C_{\mathbf{j}}\right) && (5.4.16)\\ &= \mu(\pi^{-1}(F_{i_1\cdots i_{n+1}})) && (5.4.17) \end{aligned}$$

When $i_{n+1} = 2, 3$, (5.4.14) comes from our definition of m. (5.4.15) follows from the inductive assumption. Since the $C_{j_1\cdots j_n}$ cubes are the only objects that carry any mass in the pre-image of $F_{i_1\cdots i_n}$, we get (5.4.16). (5.4.17) follows directly from (5.4.13).

This proves (5.4.1). Thus, we also have

$$m(A) = \mu(\pi^{-1}(A)) \text{ for all } A \in \mathcal{A}$$

which, according to Carathéodory's Uniqueness Theorem (Theorem 1.1.15), shows that

$$m(B) = \mu(\pi^{-1}(B)) \text{ for all } B \in \sigma(\mathcal{A}). \qquad (5.4.18)$$

□

Lemma 5.4.2. $f(x) = m([0, x]) \geq \frac{1}{4}x$ *for all* $x \in [0, \sqrt{3}\,]$.

Proof. A graph of $f(x) = m([0, x])$ can be seen in Figure 5.4.1. We prove this result by dividing into three distinct cases. The last case is slightly more difficult to prove than the first two cases:

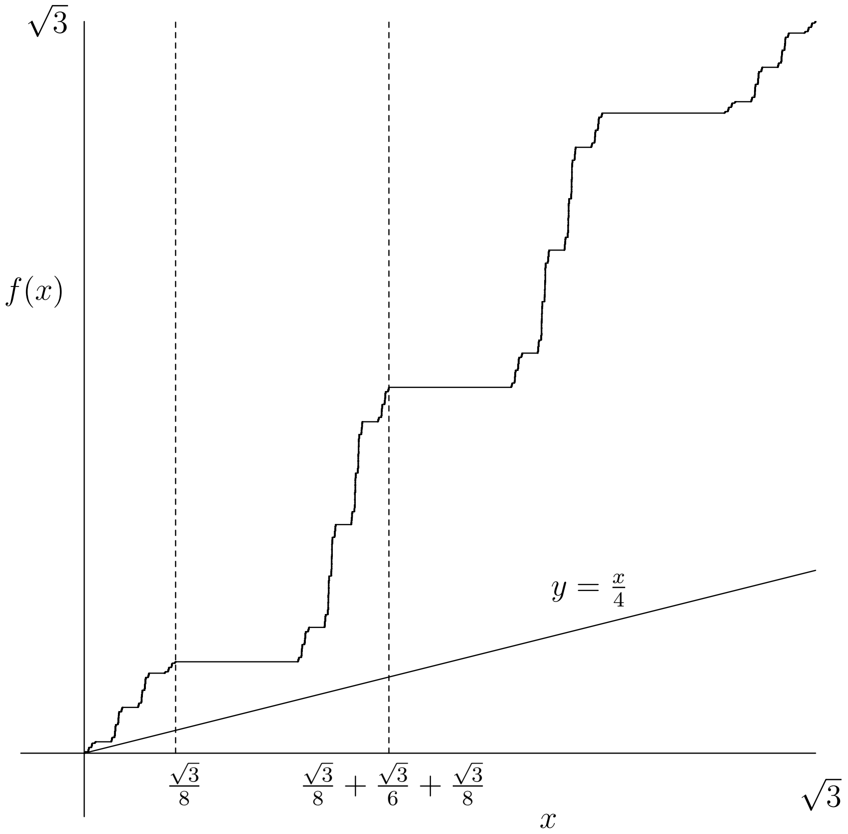


Figure 5.4.1: A graph of $f(x) = m([0, x])$ and the line $y = \frac{1}{4}x$ when $x \in [0, \sqrt{3}]$. The intervals used in each of the cases in the proof of Lemma 5.4.2 are also shown.

Case 1: $x \in \left[\frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{6} + \frac{\sqrt{3}}{8}, \sqrt{3}\right]$

We have:

$$\begin{aligned} f(x) &\geq f\left(\frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{6} + \frac{\sqrt{3}}{8}\right) \qquad (5.4.19) \\ &= \frac{\sqrt{3}}{8} + \frac{3\sqrt{3}}{8} \\ &= \frac{\sqrt{3}}{2} \\ &\geq \frac{1}{4}x. \end{aligned}$$

We get (5.4.19) by noting that f is a monotonic increasing function.

Case 2: $x \in \left[\frac{\sqrt{3}}{8}, \frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{6} + \frac{\sqrt{3}}{8}\right]$

We have:

$$\begin{aligned} f(x) &\geq f\left(\frac{\sqrt{3}}{8}\right) \\ &= \frac{\sqrt{3}}{8} \\ &\geq \frac{1}{4}\left(\frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{6} + \frac{\sqrt{3}}{8}\right) \\ &\geq \frac{1}{4}x. \end{aligned} \tag{5.4.20}$$

As in the previous case, we get (5.4.20) because f is monotonic increasing.

Case 3: $x \in \left[0, \frac{\sqrt{3}}{8}\right]$

Note that $\left[0, \frac{\sqrt{3}}{8}\right] = \bigcup_n S_1^n \left(\left[\frac{\sqrt{3}}{8}, \sqrt{3}\right]\right)$. We would like to show that

$$f(x) \geq \frac{1}{4}x \text{ when } x \in S_1^n \left(\left[\frac{\sqrt{3}}{8}, \sqrt{3}\right]\right) \text{ for all } n \geq 1.$$

We do this for $n = 1$, then prove by induction. First we show that f in the interval $[0, \frac{\sqrt{3}}{8}]$ is an S_1 re-scaling of f in the interval $[0, \sqrt{3}]$. Recall that $S_1(x) = \frac{1}{8}x$ and $S_1^{-1}(x) = 8x$ and note that $f(\frac{\sqrt{3}}{8}) = \frac{\sqrt{3}}{8}$. We want to show that $f(x) = S_1(f(8x))$ or that $f(x) = \frac{1}{8}f(8x)$ for all $x \in [0, \frac{\sqrt{3}}{8}]$. Letting $x \in [0, \frac{\sqrt{3}}{8}]$ we have:

$$f(x) = m([0, x])$$

$$\begin{aligned} &= \sum_{i=1}^{4} p_i \, m(S_i^{-1}([0,x])) \\ &= p_1 \, m(S_1^{-1}([0,x])) \qquad (5.4.21) \\ &= \frac{1}{8} \, m([0,8x]) \\ &= \frac{1}{8} \, f(8x). \end{aligned}$$

Equation (5.4.21) comes as a direct result of S_1 being the only map that maps to the interval $[0, \frac{\sqrt{3}}{8}]$. Clearly the line $\frac{1}{4}x$ in the interval $[0, \sqrt{3}]$ rescales to $\frac{1}{4}x$ in $[0, \frac{\sqrt{3}}{8}]$ under S_1 because for all $x \in [0, \frac{\sqrt{3}}{8}]$,

$$\begin{aligned} \frac{1}{4}x &= S_1(\tfrac{1}{4}.8x) \\ &= \frac{1}{8}.\frac{1}{4}.8x. \end{aligned}$$

So $f(x) \geq \frac{1}{4}x$ holds for all $x \in S_1([\frac{\sqrt{3}}{8}, \sqrt{3}]) = [\frac{\sqrt{3}}{64}, \frac{\sqrt{3}}{8}]$. We can now show that the inequality $f(x) \geq \frac{1}{4}x$ is also valid in the interval $[0, \frac{\sqrt{3}}{64}]$ by starting the induction. We assume that $f(x) \geq \frac{1}{4}x$ for all $x \in S_1^n([\frac{\sqrt{3}}{8}, \sqrt{3}])$ for some $n \geq 1$ and prove it for $n+1$. So we assume that

$$f(x) \geq \frac{1}{4}x \text{ for all } x \in S_1^n\left(\left[\tfrac{\sqrt{3}}{8}, \sqrt{3}\right]\right)$$

and aim to prove that

$$f(x) \geq \frac{1}{4}x \text{ for all } x \in S_1^{n+1}\left(\left[\tfrac{\sqrt{3}}{8}, \sqrt{3}\right]\right).$$

Let $x \in S_1^{n+1}([\frac{\sqrt{3}}{8}, \sqrt{3}])$. We have

$$f(x) = m([0,x])$$

$$
\begin{aligned}
&= \sum_{i=1}^{4} p_i \, m(S_i^{-1}([0,x])) \\
&= p_1 \, m(S_1^{-1}([0,x])) \\
&= \frac{1}{8} \, m([0, S_1^{-1}(x)]) \\
&= \frac{1}{8} \, f(S_1^{-1}(x)). \qquad (5.4.22)
\end{aligned}
$$

Since $S_1^{-1}(x) \in S_1^n([\frac{\sqrt{3}}{8}, \sqrt{3}])$ and our inductive assumption states that $f(x) \geq \frac{1}{4}x$ for all $x \in S_1^n([\frac{\sqrt{3}}{8}, \sqrt{3}])$, using (5.4.22) we conclude:

$$
\begin{aligned}
f(x) &= \frac{1}{8} \, f(S_1^{-1}(x)) \\
&\geq \frac{1}{8} \cdot \frac{1}{4} \, S_1^{-1}(x) \\
&= \frac{1}{8} \cdot \frac{1}{4} \cdot 8x \\
&= \frac{1}{4} \, x.
\end{aligned}
$$

We have shown that

$$
f(x) \geq \frac{1}{4}x \text{ for all } x \in \left[0, \tfrac{\sqrt{3}}{8}\right] \cup \left[\tfrac{\sqrt{3}}{8}, \tfrac{\sqrt{3}}{8} + \tfrac{\sqrt{3}}{6} + \tfrac{\sqrt{3}}{8}\right] \cup \left[\tfrac{\sqrt{3}}{8} + \tfrac{\sqrt{3}}{6} + \tfrac{\sqrt{3}}{8}, \sqrt{3}\right]
$$

thus completing the proof. □

Proposition 5.4.3. $\mu(\triangle x) \geq \frac{1}{4}x$

Proof. We can easily prove this using Lemma 5.4.1 and Lemma 5.4.2. Given a $\triangle x$,

$$
\mu(\triangle x) = \mu(\pi^{-1}([0,x]))
$$

where $[0, x] \subseteq F$. Using the two lemmas, we have

$$\mu(\pi^{-1}([0, x])) = m([0, x]) \geq \frac{1}{4}x$$

□

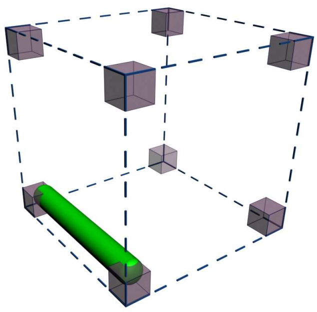
5.5 Proof of Main Result

Recall Theorem 5.2.1 which says that $\mathcal{H}^1(C) = \sqrt{3}$. We now prove this result.

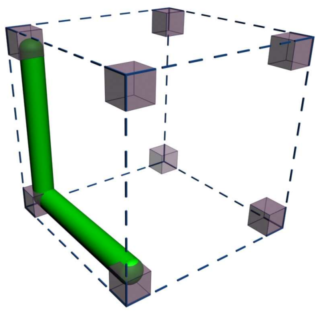
Proof. We start with the upper bound:

"$\leq$" This follows from Theorem 4.5.1.

"$\geq$" According to the mass distribution principle, if $\mu(V) \leq |V|$ for all measurable sets V, then $\mathcal{H}^1(C) \geq \mu(C)$.



(a) Case 1.1



(b) Case 1.2

Figure 5.5.1: A set V is shown intersecting exactly 2 C_j cubes at the first level of the construction of a Sierpinski sponge in (a), and in (b), V intersects exactly 3 C_j cubes. While there are other possible configurations, the two shown above assist our calculations because V intersects particular cubes that provide the lowest possible diameter for V.

<u>Case 1:</u> V intersects $2, 3, 4, 5$ or 6 of the C_j cubes.

The method for verifying that $\mu(V) \leq |V|$ in each of these situations is similar, so we group them all under one main case here.

Case 1.1: V intersects exactly 2 of the C_j cubes.

$$|V| \geq \frac{6}{8} > \frac{2\sqrt{3}}{8} \geq \mu(V).$$

Case 1.2: V intersects exactly 3 of the C_j cubes.

$$|V| \geq \frac{6}{8} > \frac{3\sqrt{3}}{8} \geq \mu(V).$$

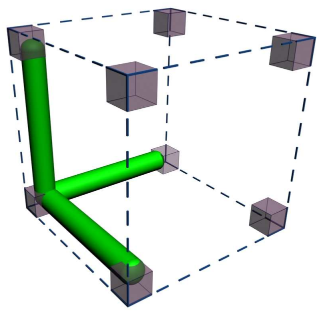


Figure 5.5.2: Case 1.3: V intersects exactly 4 of the C_j cubes.

Case 1.3: V intersects exactly 4 of the C_j cubes.

$$\begin{aligned} |V| \quad &\geq \quad \sqrt{2} - \frac{2}{8}\sqrt{2} = \frac{6}{8}\sqrt{2} = \frac{\sqrt{3}^2\sqrt{2}^3}{8} = \frac{\sqrt{3}\sqrt{3}\sqrt{2}^3}{8} \\ &> \quad \frac{\sqrt{3}\sqrt{2}\sqrt{2}^3}{8} = \frac{4\sqrt{3}}{8} \geq \mu(V). \end{aligned}$$

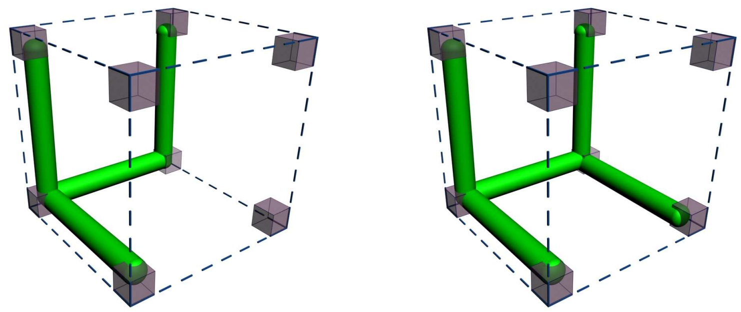
Case 1.4: V intersects exactly 5 of the C_j cubes.

$$|V| \geq \sqrt{3} - \frac{2}{8}\sqrt{3} = \frac{6}{8}\sqrt{3} > \frac{5}{8}\sqrt{3} \geq \mu(V).$$

Case 1.5: V intersects exactly 6 of the C_j cubes.

$$|V| \geq \sqrt{3} - \frac{2}{8}\sqrt{3} = \frac{6}{8}\sqrt{3} \geq \mu(V).$$

<u>Case 2:</u> V intersects 7 or 8 of the C_j cubes.



(a) Case 1.4 (b) Case 1.5

Figure 5.5.3: Cases 1.4 and 1.5: V intersects exactly 5 of the C_j cubes on the left of this figure and V intersects exactly 6 C_j cubes on the right.

This main case breaks down into two subcases which are, much like the ones in the preceding main case, dealt with in a similar way. For the purpose of laying out some notational details, we will assume momentarily that V intersects exactly 8 of the C_j.

Given a C_j, let u_j be the vertex C_j shares with $C_\emptyset$ and let w_j be the vertex of C_j that is not a member of $\partial C_\emptyset$. Let G_j denote the plane that is perpendicular to the line that passes through u_j and w_j, and that passes through w_j. Let A_j denote the plane that is perpendicular to the line that passes through u_j and w_j, that intersects the boundary, ∂V, of V and that is parallel to G_j. Let $a_j = d(u_j, A_j)$ and $g_j = d(G_j, A_j)$. It's clear that $a_j + g_j = \frac{\sqrt{3}}{8}$ for all j and that

$$|V| \geq \frac{6}{8}\sqrt{3} + g_1 + g_4, \quad (5.5.1)$$

$$|V| \geq \frac{6}{8}\sqrt{3} + g_{2_1} + g_{3_2}, \quad (5.5.2)$$

$$|V| \geq \frac{6}{8}\sqrt{3} + g_{2_2} + g_{3_1}, \quad (5.5.3)$$

$$|V| \geq \frac{6}{8}\sqrt{3} + g_{2_3} + g_{3_3}. \quad (5.5.4)$$

Having established the necessary notation, we now proceed to the two subcases.

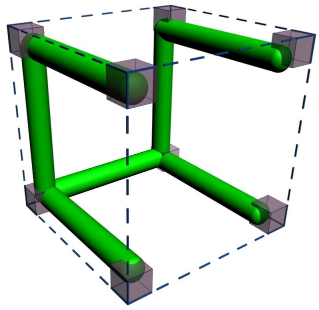


Figure 5.5.4: Case 2.1: V intersects exactly 8 of the C_j cubes.

Case 2.1: V intersects exactly 8 of the C_j cubes.

By adding Equations (5.5.1), (5.5.2), (5.5.3) and (5.5.4), we get

$$\begin{aligned} 4|V| &\geq 4.\frac{6}{8}\sqrt{3} + g_1 + g_{2_1} + g_{2_2} + g_{2_3} + g_{3_1} + g_{3_2} + g_{3_3} + g_4 \\ |V| &\geq \frac{6}{8}\sqrt{3} + \frac{1}{4}(g_1 + g_{2_1} + g_{2_2} + g_{2_3} + g_{3_1} + g_{3_2} + g_{3_3} + g_4) \\ &= \frac{6}{8}\sqrt{3} + \frac{1}{4}(\sqrt{3} - (a_1 + a_{2_1} + a_{2_2} + a_{2_3} + a_{3_1} + a_{3_2} + a_{3_3} + a_4) \\ &= \sqrt{3} - \frac{1}{4}(a_1 + a_{2_1} + a_{2_2} + a_{2_3} + a_{3_1} + a_{3_2} + a_{3_3} + a_4) \qquad (5.5.5) \end{aligned}$$

By Lemma 5.4.3 and Equation (5.5.5), we have:

$$\begin{aligned} \mu(V) &\leq \sqrt{3} - (\mu(\triangle a_1) + \mu(\triangle a_{2_1}) + \mu(\triangle a_{2_2}) + \mu(\triangle a_{2_3}) + \\ &\quad \mu(\triangle a_{3_1}) + \mu(\triangle a_{3_2}) + \mu(\triangle a_{3_3}) + \mu(\triangle a_4)) \\ &\leq \sqrt{3} - \frac{1}{4}(a_1 + a_{2_1} + a_{2_2} + a_{2_3} + a_{3_1} + a_{3_2} + a_{3_3} + a_4) \\ &\leq |V|. \end{aligned}$$

Case 2.2: V intersects exactly 7 of the C_j cubes.

Without loss of generality, we can assume that V does not intersect C_4. Using the same notation, but disregarding Equation (5.5.1), we may now add Equations

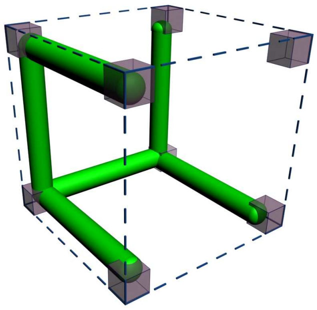


Figure 5.5.5: Case 2.2: V intersects exactly 7 of the C_j cubes.

(5.5.2), (5.5.3) and (5.5.4) to get:

$$\begin{aligned} 3|V| &\geq 3.\frac{6}{8}\sqrt{3} + g_{2_1} + g_{2_2} + g_{2_3} + g_{3_1} + g_{3_2} + g_{3_3} \\ |V| &\geq \frac{6}{8}\sqrt{3} + \frac{1}{3}(g_{2_1} + g_{2_2} + g_{2_3} + g_{3_1} + g_{3_2} + g_{3_3}) \\ &= \frac{6}{8}\sqrt{3} + \frac{1}{3}(\sqrt{3} - (a_{2_1} + a_{2_2} + a_{2_3} + a_{3_1} + a_{3_2} + a_{3_3}) \\ &= \sqrt{3} - \frac{1}{3}(a_{2_1} + a_{2_2} + a_{2_3} + a_{3_1} + a_{3_2} + a_{3_3}) \end{aligned} \tag{5.5.6}$$

By Lemma 5.4.3 and Equation (5.5.6) above, we have:

$$\begin{aligned} \mu(V) &\leq \frac{7}{8}\sqrt{3} - (\mu(\triangle a_1) + \mu(\triangle a_{2_1}) + \mu(\triangle a_{2_2}) + \mu(\triangle a_{2_3}) + \\ &\quad \mu(\triangle a_{3_1}) + \mu(\triangle a_{3_2}) + \mu(\triangle a_{3_3})) \\ &\leq \frac{7}{8}\sqrt{3} - \frac{1}{4}(a_1 + a_{2_1} + a_{2_2} + a_{2_3} + a_{3_1} + a_{3_2} + a_{3_3}). \end{aligned} \tag{5.5.7}$$

But subtracting (5.5.7) from (5.5.6) we get:

$$\begin{aligned} |V| - \mu(V) &\geq \sqrt{3} - \frac{1}{3}(a_{2_1} + a_{2_2} + a_{2_3} + a_{3_1} + a_{3_2} + a_{3_3}) \\ &\quad -\frac{7}{8}\sqrt{3} + \frac{1}{4}(a_1 + a_{2_1} + a_{2_2} + a_{2_3} + a_{3_1} + a_{3_2} + a_{3_3}) \\ &= \frac{1}{8}\sqrt{3} - \frac{1}{12}(a_{2_1} + a_{2_2} + a_{2_3} + a_{3_1} + a_{3_2} + a_{3_3}) + \frac{1}{4}a_1 \\ &\geq \frac{1}{8}\sqrt{3} - \frac{1}{12}.\frac{6}{8}\sqrt{3} \\ &= \frac{4}{32}\sqrt{3} - \frac{2}{32}\sqrt{3} \quad \geq \quad 0. \end{aligned}$$

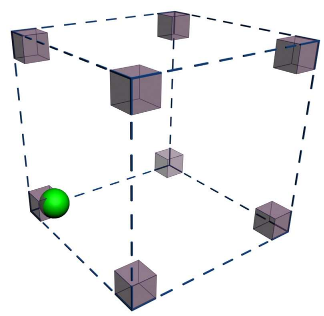


Figure 5.5.6: Case 3: V intersects exactly 1 C_j cube.

<u>Case 3:</u> V intersects exactly 1 C_j cube.

We divide this into 2 distinct subcases:

Case 3.1: V intersects 2, 3, 4, 5, 6, 7 or 8 $C_{j_1 j_2}$ cubes.

Case 3.2: V intersects exactly 1 $C_{j_1 j_2}$ cube.

Proving Case 3.1 simply requires a repeat of the proofs in case 1 and case 2 over C_j instead of $C_\emptyset$. Case 3.2 requires that we divide it into a further 2 subcases where V intersects either 2, 3, 4, 5, 6, 7 or 8 $C_{j_1 j_2 j_3}$ cubes or exactly one $C_{j_1 j_2 j_3}$ cube.

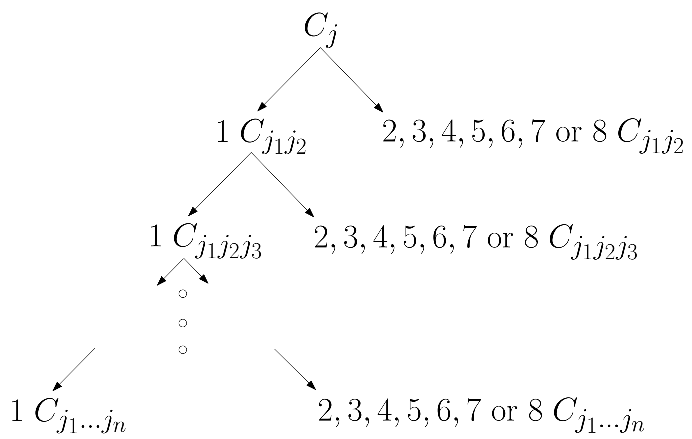


Figure 5.5.7: A tree representation of the proof of Case 3.

The cases where V intersects 2, 3, 4, 5, 6, 7 or 8 $C_{j_1\cdots j_n}$ cubes can be proven for all n by repeating the proofs for Case 1 and Case 2. When V intersects exactly one $C_{j_1\cdots j_n}$ for all n, we have

$$V \subseteq \bigcup_n C_{j_1\cdots j_n} = \{x\} \, .$$

So $\mu(V) = 0 \leq |V|$ and we are done. A tree structure of these subcases is shown in Figure 5.5.7.

□

Chapter 6

Further Directions

In this chapter we take a brief look at some further directions in which the work discussed up until now could taken. As was mentioned in Chapter 2, during the course of his studies, the author became particularly interested in iterated function systems with condensation. In the next section, we take a look at how the Hausdorff measure behaves when measuring such sets. In the subsequent section, we review some of the work of Zhou, Zhu and Luo on packing measure and discuss how the packing measure of the Sierpinski carpet and Sierpinski sponge might be calculated.

6.1 Iterated Function Systems with Condensation and the Hausdorff Measure

Let us consider the set shown in Figure 6.1.1 which is in a complete metric space (X, d) in $\mathbb{R}^2$. This set is generated by the IFS used to generate the Sierpinski carpet in Chapter 4, as well as a condensation set which is similar to the Sierpinski carpet from Chapter 4 and $\frac{1}{4}$ of its size. The condensation set is located in the centre of the set and

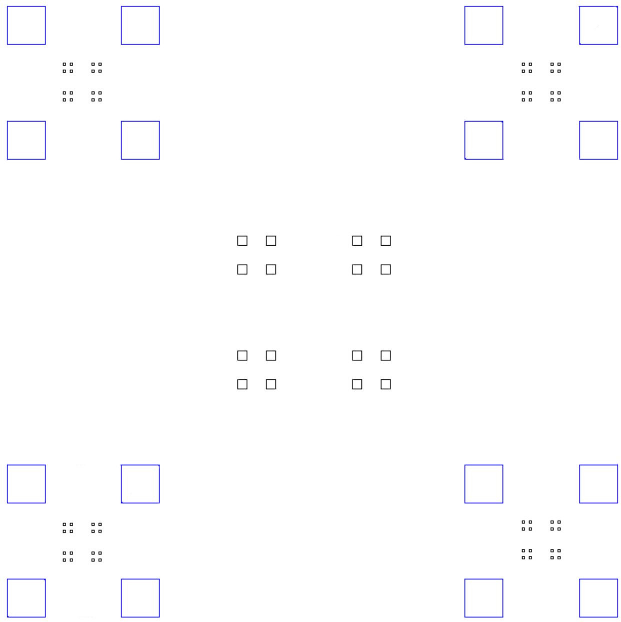


Figure 6.1.1: This diagram shows the first two levels of the construction of a Sierpinski carpet with a condensation set which is $\frac{1}{4}$ the size of a regular Sierpinski carpet (as described in Chapter 4). Note that only the first two levels of the construction of the condensation set are shown.

smaller copies of it are transformed into the four corners of the image under the action of the IFS. In the following, we will label the original carpet K. Assuming that the IFS used to generate the original carpet is $\{S_1, S_2, S_3, S_4\}$ with contraction ratios $\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\}$, then if we label the condensation set C, the invariant set for the IFS mixed with C is given by

$$K_c = C \cup \left(\bigcup_{i=1}^{4} S_i(K_c) \right).$$

Looking at the K_c set in the figure, the following heuristic calculation of its Hausdorff measure seems reasonable:

$$\begin{aligned}
\mathcal{H}^1(K_c) &= \mathcal{H}^1(K) + \mathcal{H}^1(C) + 4\,\mathcal{H}^1\left(\frac{1}{4}\,C\right) + 16\,\mathcal{H}^1\left(\frac{1}{16}\,C\right) + \cdots \\
&= \sqrt{2} + \mathcal{H}^1\left(\frac{1}{4}K\right) + \mathcal{H}^1\left(\frac{1}{4}K\right) + \mathcal{H}^1\left(\frac{1}{4}K\right) + \cdots \\
&= \sqrt{2} + \frac{1}{4}\mathcal{H}^1(K) + \frac{1}{4}\mathcal{H}^1(K) + \frac{1}{4}\mathcal{H}^1(K) + \cdots \\
&= \sqrt{2} + \frac{1}{4}\sqrt{2} + \frac{1}{4}\sqrt{2} + \frac{1}{4}\sqrt{2} + \cdots \\
&= \sqrt{2}\left(1 + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \cdots\right) \\
&= +\infty.
\end{aligned}$$

We can generalise the calculation as follows. Let $\{S_i\}_m$ be some IFS in a complete metric space (X, d) in $\mathbb{R}^n$ and let $\{c_i\}_m$ be its associated contraction ratios. We will label the invariant set generated by the IFS K. We will assume that the open set condition holds so that K has positive finite Hausdorff measure at the critical dimension. Given some non-empty compact set C in the metric space, the invariant set generated by the IFS mixed with C in the usual way is labelled K_c. We will also assume that $0 < \mathcal{H}^{\dim_{\mathcal{H}} C}(C) < \infty$. We may derive the following using the standard properties of iterated function systems:

$$\begin{aligned}
K_c &= \left(\bigcup_{i=1}^{m} S_i(K_c)\right) \cup\, C \\
&= \left(\bigcup_{i=1}^{m} S_i\left(\left(\bigcup_{i=1}^{m} S_i(K_c)\right) \cup\, C\right)\right) \cup\, C \\
&= \left(\bigcup_{i=1}^{m} S_i\left(\bigcup_{i=1}^{m} S_i(K_c)\right)\right) \cup\, \left(\bigcup_{i=1}^{m} S_i(C)\right) \cup\, C \\
&= \left(\bigcup_{i=1}^{m} S_i\left(\bigcup_{i=1}^{m} S_i\left(\bigcup_{i=1}^{m} S_i(K_c)\right)\right)\right) \cup \\
&\quad \left(\bigcup_{i=1}^{m} S_i\left(\bigcup_{i=1}^{m} S_i(C)\right)\right) \cup\, \left(\bigcup_{i=1}^{m} S_i(C)\right) \cup\, C
\end{aligned}$$

$$\begin{aligned}
&= \left(\bigcup_{i=1}^{m} S_i\left(\bigcup_{i=1}^{m} S_i\left(\bigcup_{i=1}^{m} S_i\left(\cdots(K_c)\right)\right)\right)\right) \cup \\
&\quad C \cup \left(\bigcup_{i=1}^{m} S_i(C)\right) \cup \left(\bigcup_{i=1}^{m} S_i\left(\bigcup_{i=1}^{m} S_i(C)\right)\right) \cup \cdots \\
&= K \cup C \cup \left(\bigcup_{i=1}^{m} S_i(C)\right) \cup \left(\bigcup_{i=1}^{m} S_i\left(\bigcup_{i=1}^{m} S_i(C)\right)\right) \cup \cdots
\end{aligned}$$

As is highlighted by Falconer in [Fal90], the Hausdorff dimension is stable under countable unions, so since the S_i mappings transform C to similar copies of itself, it is safe to say that

$$s = \dim_{\mathcal{H}} K_c = \max\left\{\dim_{\mathcal{H}} K, \dim_{\mathcal{H}} C\right\}.$$

Assuming the $S_{\mathbf{i}}(C)$ are disjoint, taking the Hausdorff measure of both sides and using both the countable additivity and scaling properties of Hausdorff measure we get:

$$\begin{aligned}
\mathcal{H}^s(K_c) &= \mathcal{H}^s(K) + \mathcal{H}^s(C) + \sum_{i=1}^{m} \mathcal{H}^s(S_i(C)) + \sum_{j=1}^{m}\sum_{i=1}^{m} \mathcal{H}^s(S_j(S_i(C))) + \cdots \\
&= \mathcal{H}^s(K) + \mathcal{H}^s(C) + \sum_{i=1}^{m} c_i^s \mathcal{H}^s(C) + \sum_{j=1}^{m}\sum_{i=1}^{m} c_j^s c_i^s \mathcal{H}^s(C) + \cdots \\
&= \mathcal{H}^s(K) + \mathcal{H}^s(C)\left(1 + \sum_{i=1}^{m} c_i^s + \sum_{j=1}^{m}\sum_{i=1}^{m} c_i^s c_j^s + \cdots\right) \\
&= \mathcal{H}^s(K) + \mathcal{H}^s(C)\left(1 + \left(\sum_{i=1}^{m} c_i^s\right)^1 + \left(\sum_{i=1}^{m} c_i^s\right)^2 + \cdots\right).
\end{aligned}$$

Owing to the definition of Hausdorff dimension (Definition 1.3.6), we now have two distinct cases:

Case 1: $s = \dim_{\mathcal{H}} K > \dim_{\mathcal{H}} C$:

$$\begin{aligned} \mathcal{H}^s(K_c) &= \mathcal{H}^s(K) + \mathcal{H}^s(C)\left(1 + \left(\sum_{i=1}^{m} c_i^s\right)^1 + \left(\sum_{i=1}^{m} c_i^s\right)^2 + \cdots\right) \\ &= \mathcal{H}^s(K) + 0\left(1 + \left(\sum_{i=1}^{m} c_i^s\right)^1 + \left(\sum_{i=1}^{m} c_i^s\right)^2 + \cdots\right) \\ &= \mathcal{H}^s(K). \end{aligned}$$

Case 2: $s = \dim_{\mathcal{H}} C \geq \dim_{\mathcal{H}} K$:

$$\mathcal{H}^s(K_c) = \mathcal{H}^s(K) + \mathcal{H}^s(C)\left(1 + \left(\sum_{i=1}^{m} c_i^s\right)^1 + \left(\sum_{i=1}^{m} c_i^s\right)^2 + \cdots\right)$$

Looking at the second case, which is the more interesting of the two, clearly the infinite sum in brackets is a geometric series, so if $(\sum_{i=1}^{m} c_i^s) < 1$, the sum converges and we have a positive finite value for $\mathcal{H}^s(K_c)$. If however $(\sum_{i=1}^{m} c_i^s) \geq 1$, the sum diverges and we are left with $\mathcal{H}^s(K_c) = +\infty$.

This leaves us in a rather puzzling situation. If we take the set generated by the IFS with condensation from Figure 6.1.1 where $s = \dim_{\mathcal{H}} K = \dim_{\mathcal{H}} C = 1$ and construct a similar set using the same condensation set, but an IFS with slightly smaller contraction ratios (0.249999995 say, as opposed to $\frac{1}{4}$), then while the former set retains its Hausdorff measure of $+\infty$, the latter set will have positive finite Hausdorff measure. Similarly, if we were to take the original IFS with condensation and remove one of the four similarity mappings from the IFS so that we have three similarities in the IFS, each with a contraction ratio of $\frac{1}{4}$, the invariant set generated by this new IFS with condensation would have positive finite Hausdorff measure. The original set generated by the IFS with condensation is not hugely different to the sets generated by the two modified examples. Each of the three sets is clearly a fractal. This begs some interesting questions. Should the Hausdorff measure differ so markedly between such similar sets? Based on the above calculation, it is clear that there exists an extremely large class of fractal sets which have Hausdorff measure of $+\infty$. Would it be possible to construct a modified version of the Hausdorff measure which would assign a positive finite value to such sets? Perhaps these questions could form a good basis for some future work.

6.2 Packing Measure and Dimension

The packing measure and dimension are considered to be of equal importance to the Hausdorff measure and dimension in the modern world of fractal geometry. Making their first appearance in the 1980's in papers by Tricot [Tri82], Taylor & Tricot [TT85] and Raymond & Tricot [RT88], they are similar to the Hausdorff measure and dimension, but use efficient *packings* of small balls instead of efficient *coverings* of small balls in their definition. As Falconer points out in [Fal90], given that the Hausdorff dimension extends the basic premise of the lower box counting dimension $\underline{\dim}_B$ by utilising efficient coverings of balls of differing radii as opposed to balls of equal radii, it is natural to try to extend the idea behind the upper box counting dimension $\overline{\dim}_B$ in a similar way, so that dense packings of disjoint balls of differing radii are used instead of disjoint balls of equal radii. This is precisely what is attempted with the packing dimension, which requires that we derive a suitable notion of packing measure first.

6.2.1 Definitions

The following definition of δ-approximative pre-packing measure is structured in a similar way to the definition of δ-approximative Hausdorff measure, but uses dense packings of disjoint balls instead of economical coverings of small balls. Note that the term *centered* δ*-packing* of some set E in a metric space X refers to a countable family of closed balls in X with centres in E and radii at most δ.

Definition 6.2.1. We define the δ-approximative s-dimensional pre-packing measure of a set $E \subseteq X$, where X is a metric space, as:

$$\overline{\mathcal{P}}^s_\delta(E) = \sup\left\{\sum_{i=1}^{\infty} \text{diam}(B_i)^s : \{B_i\}_i \text{ is a centered } \delta\text{-packing of E}\right\}.$$

In a similar way as with the Hausdorff measure, we seek the limit of $\mathcal{P}^s_\delta$ as δ tends to zero and define the pre-packing measure as follows:

Definition 6.2.2. Letting E be defined as in the previous definition, the pre-packing measure of E is:

$$\overline{\mathcal{P}}^s(E) = \underline{\lim}_{\delta\to 0} \overline{\mathcal{P}}^s_\delta(E).$$

Unfortunately, as was illustrated by Taylor and Tricot in [TT85], $\mathcal{P}^s$ is not necessarily countable subadditive and so, not necessarily a measure. However, we can modify the above definition to something that can be shown to be a Borel measure. We call this the packing measure and it is defined below.

Definition 6.2.3. Letting E be defined as in the previous definitions, the s-dimensional packing measure of E is:

$$\mathcal{P}^s(E) = \inf\left\{\sum_{i}^{\infty} \overline{\mathcal{P}}^s(E_i) : E \subseteq \bigcup_{i=1}^{\infty} E_i\right\}.$$

This conveniently leads us to a definition of packing dimension which, again, is similar to the Hausdorff definition of dimension:

Definition 6.2.4. The packing dimension of the set E as defined in the above is given by

$$\dim_p E = \sup\{s : \mathcal{P}^s(E) = \infty\} = \inf\{s : \mathcal{P}^s(E) = 0\}.$$

It is well known that $\dim_{\mathcal{H}} E \leq \dim_P E \leq \overline{\dim}_B E$. A proof of this may be found in [Fal90].

6.2.2 Packing measure of Sierpinski sets

In [JZZL03] and [JZ04], the authors Jia, Zhou, Zhu & Luo and Jia & Zhu respectively, calculate the packing measure of Sierpinski sets in the plane which are similar to the one we analysed in Chapter 4. The authors of [JZZL03] develop a technique for calculating the packing measure of the Cartesian product of the middle-third Cantor set with itself, but their method can be generalised for other similar Sierpinski carpets under certain conditions. A paper [ZZL04] due to Zhu, Zhou and Luo also exists which analyses the packing measure of a class of generalised Sierpinski sponges, but unfortunately a suitable translation could not be obtained at the time of writing.

In this section, we will take a look at the result garnered by Jia *et al* in [JZZL03] and sketch its proof. The main result is as follows:

Theorem 6.2.5. *The packing measure of the Cartesian product of the middle third Cantor set with itself, labelled $C \times C$ is as follows:*

$$\mathcal{P}^{\log_3 4}(C \times C) = 4^{\log_3 4}.$$

The full proof of this result, incorporating the proofs of a number of necessary lemmas, is too lengthy to be fully dissected here, so we will try to attain a broad overview of the main problem and analyse the key lemmas in more detail. We start with some notation:

We refer to the middle-third Cantor set in the unit interval as C and the Cartesian product of two such Cantor sets as $C \times C$. Establishing an orthogonal coordinate system in $\mathbb{R}^2$, we define $E_0 = [0,1] \times [0,1]$ which shares a vertex with the origin, and an IFS $\{f_1, f_2, f_3, f_4\}$ such that the IFS acting on E_0 yields $C \times C$ which, naturally, is invariant under the action of the IFS. The proof from [JZZL03] requires that the IFS satisfies a stronger version of the open set condition known as the *strong separation condition* which we now define:

Definition 6.2.6. Given an IFS $\{S_1, \dots, S_n\}$ in some metric space, the *strong separation condition* is satisfied if $S_i(E) \cap S_j(E) = \emptyset$ for all i, j with $i \neq j$.

The specific IFS mappings are important to us in the proof, so we define them as follows:

$$f_i(x) = \frac{x}{3} + b_i \quad \text{where} \quad i = \{1,2,3,4\}, x \in \mathbb{R}^2,$$

where $b_1 = (0,0), b_2 = (\frac{2}{3}, 0), b_3 = (\frac{2}{3}, \frac{2}{3}), b_4 = (0, \frac{2}{3})$. So f_1 maps to the bottom-left of E_0, f_2 maps to the bottom-right, f_3 maps to the top-right and f_4 maps to the top-left. Clearly we have

$$C \times C = \bigcup_{n=1}^{4} f_n(C \times C).$$

The term *basic square of the nth level* is used in a similar way as in previous chapters. For instance, $f_1(E_0)$ is a basic square of the first level of the construction of $C \times C$, $f_2(f_1(E_0))$ is a basic square of the second level of the construction and so on. Incidentally, the phrases *"nth level of the construction of $C \times C$"* and *"nth iteration of the IFS over E_0"* are interchangeable in the current context. For any integer $k \geq 0$, define

$$\begin{aligned} I_k &= \{(i_1, i_2, i_3, \dots, i_k) : i_j \in \{1,2,3,4\}, j = 1, 2, \dots, k\}, \\ I_\infty &= \{(i_1, i_2, i_3, \dots) : i_j \in \{1,2,3,4\}, j = 1, 2, \dots\}. \end{aligned}$$

For any $k \geq 1$, let

$$f_1^k = f_{1_1} \circ f_{1_2} \circ f_{1_3} \circ \cdots \circ f_{1_k}.$$

The following notation is used to define the union of all basic squares from level $p+k$ onwards that lie in the bottom-left-most basic square at the kth level of the construction of $C \times C$ (i.e. the kth iteration of the IFS over E_0).

$$F_{p,k} = \bigcup_{n=p}^{\infty} \bigcup_{(i_1 i_2 \ldots i_n) \in I_n} f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n} \left(f_1^k(E_0) \right) \qquad (p \geq 1). \tag{6.2.1}$$

We refer to the packing dimension of the set $C \times C$ as

$$s = \dim_{\mathcal{P}}(C \times C) = \log_3 4.$$

This is a well known result.

As always, we shall refer to a ball of radius r centered at a point x as $B_r(x)$. However, we may also use the notation $B(x, r)$ interchangeably to mean the same thing.

This first lemma is the packing measure analogue of Proposition 3.2.5 introduced in Chapter 3.

Lemma 6.2.7. *Let $E \subset \mathbb{R}^m$ be a Borel set, μ be a finite Borel measure, $0 < c < \infty$.*

(a) If $\displaystyle\underline{\lim}_{r \to 0} \frac{\mu(B_r(x))}{r^s} \leq c$ for all $x \in E$, then $\mathcal{P}^s(E) \geq 2^s \dfrac{\mu(E)}{c}$.

(b) If $\displaystyle\underline{\lim}_{r \to 0} \frac{\mu(B_r(x))}{r^s} \geq c$ for all $x \in E$, then $\mathcal{P}^s(E) \leq 2^s \dfrac{\mu(E)}{c}$.

Proof. Omitted. □

Jia *et al* proceed by defining a self-similar measure μ with support $C \times C$ for any

Borel set $E \subseteq \mathbb{R}^m$. This measure acts by taking the packing measure of any given part of $C \times C$ and normalising it by the total packing measure of $C \times C$. We define the measure as follows:

$$\mu(E) = \frac{\mathcal{P}^s(E \cap (C \times C))}{\mathcal{P}^s(C \times C)} \tag{6.2.2}$$

The previous lemma (6.2.7) tells us that the packing measure of $C \times C$ is determined by the lower spherical density of every point of $C \times C$, so we proceed by making the following two definitions for lower spherical density. The first takes the density of a point x with respect to the measure μ and the second takes the density of x with respect to the packing measure:

$$\underline{D}^s(\mu, x) = \underline{\lim}_{r \to 0} \frac{\mu(B_r(x))}{(2r)^s}, x \in \mathbb{R}^n.$$

$$\underline{D}^s(\mathcal{P}^s, x) = \underline{\lim}_{rx \to 0} \frac{\mathcal{P}^s(B_r(x))}{(2r)^s}, x \in \mathbb{R}^n.$$

Jia *et al* make use of the following lemma which is taken from [TT86] and [RT88] and is required in the proof of the subsequent lemma:

Lemma 6.2.8. *Let $\underline{D}^s(\mathcal{P}^s, x)$ be defined as above. Then*

$$\underline{D}^s(\mathcal{P}^s, x) = 1 \quad \textit{for } \mathcal{P}^s\textit{-almost all } x \in C \times C.$$

This next lemma is one of the key lemmas required for the proof of Theorem 6.2.5 and we actually proved a similar result for the Hausdorff measure in Chapter 3, namely Theorem 3.3.13.

Lemma 6.2.9. *Let μ and $\underline{D}^s(\mu, x)$ be defined as above. Then*

$$\mathcal{P}^s(C \times C) = \frac{1}{\underline{D}^s(\mu, x)} \quad \textit{for } \mu\textit{-almost all } x \in C \times C.$$

Proof. Using both the definition of $\underline{D}^s(\mu, x)$ and Lemma 6.2.8, we have:

$$\begin{aligned}
\underline{D}^s(\mu, x) &= \underline{\lim}_{r\to 0} \frac{\mu(B_r(x))}{(2r)^s} \\
&= \underline{\lim}_{r\to 0} \frac{\mathcal{P}^s(B_r(x)\cap(C\times C))}{\mathcal{P}^s(C\times C)(2r)^s} \\
&= \frac{1}{\mathcal{P}^s(C\times C)} \underline{\lim}_{r\to 0} \frac{\mathcal{P}^s(B_r(x)\cap(C\times C))}{(2r)^s} \\
&= \frac{1}{\mathcal{P}^s(C\times C)} \underline{D}^s(\mathcal{P}^s, x) \\
&= \frac{1}{\mathcal{P}^s(C\times C)}
\end{aligned}$$

for $\mathcal{P}^s$-almost all $x \in C \times C$. μ is simply a normalised version of $\mathcal{P}^s$, so this also holds for μ-almost all $x \in C \times C$. □

Lemma 6.2.10. *Let $k > 1$. Let μ and $F_{p,k}$ be defined as in* (6.2.2) *and* (6.2.1) *respectively. Then for any $p \geq 1$, $k \geq 1$ we have:*

$$\mu(F_{p,k}) = 1, \qquad \mu\left(\bigcap_{p\geq 1} F_{p,k}\right) = 1, \qquad \mu\left(\bigcap_{k\geq 1}\left(\bigcap_{p\geq 1} F_{p,k}\right)\right) = 1.$$

Proof. We omit the proof for this lemma, but it may be found in [JZZL03], [ZZL04] and [Fen03]. □

Lemma 6.2.11. *For $n \geq 0$, let V_n denote the set of all vertices of all basic squares at the nth level of the construction of $C \times C$. Then*

$$\underline{D}^s(\mu, x) = \frac{1}{4^s} \quad \text{for any } x \in V_n.$$

Proof.

"$\leq$" x is a vertex of a basic square at the nth level of the construction of $C \times C$, so this basic square must have sidelength 3^{-n}. Let $r = 3^{-n} - 3^{-(n+1)}$. Then $B_r(x)$ contains only one basic square from the $n+1$-th level of the construction. Recall that $s = \log_3 4$, so $3^s = 4$ and we have

$$\frac{\mu(B_r(x))}{(2r)^s} = \frac{4^{-(n+1)}}{(2.3^{-n} - 2.3^{-(n+1)})^s} = \frac{4^{-(n+1)}}{2^s\left(3^{-n}\left(1-\frac{1}{3}\right)\right)^s} = \frac{4^{-(n+1)}}{2^s \,.\, 4^{-n} \,.\, \frac{2^s}{3^s}} = \frac{1}{4^s}.$$

Therefore

$$\inf_{r \leq 3^{-s}} \frac{\mu(B_r(x))}{(2r)^s} \leq \frac{1}{4^s}$$

and hence

$$\underline{\lim}_{r \to 0} \frac{\mu(B_r(x))}{(2r)^s} \leq \frac{1}{4^s}.$$

"$\geq$" We will not go into the detail of the proof of this inequality, but in [JZZL03], the authors argue that since $C \times C$ is self-similar, it suffices to consider only $x \in V_0$ and $\frac{\sqrt{2}}{3} < r \leq \frac{2}{3}$. They then prove the inequality for when x is the origin in three cases: when $\frac{\sqrt{2}}{3} < r \leq \frac{2}{3}$, when $\frac{2}{3} < r \leq \sqrt{1+\frac{1}{3^2}}$ and $\sqrt{1+\frac{1}{3^2}} < r \leq \sqrt{2}$. The first and last cases follow easily from the definitions, but the middle case requires some tricky numerical calculations involving inductive and symmetrical arguments.

□

Lemma 6.2.12. *Let μ and $F_{p,k}$ be defined as in* (6.2.2) & (6.2.1) *respectively. Then*

$$\underline{D}^s(\mu, x) = \frac{1}{4^s} \quad \textit{for } \mu\textit{-almost all } x \in C \times C.$$

Proof.

"$\leq$" Let $x \in (C \times C) \cap \left(\bigcap_{k\geq 1}^{\infty} \left(\bigcap_{p\geq 1}^{\infty} F_{p,k}\right)\right)$. Then $x \in F_{p,k}$ for all $p \geq 1$ and $k \geq 1$.

Taking an integer $k \geq 1$, it is clear from the definition of $F_{p,k}$ that there exists an integer $n_p \geq p$ such that we may find $y_{n_p} \in V_{n_p}$ (where V_n is defined as in Lemma 6.2.11) with $\text{dist}(x, y_{n_p}) \leq \dfrac{\sqrt{2}}{3^{n_p+k}}$.

Taking $r_p = \dfrac{1}{3^{n_p}} - \dfrac{1}{3^{n_p+1}} - \dfrac{\sqrt{2}}{3^{n_p+k}}$, we have

$$B(x, r_p) \subset B\left(y_{n_p}, \frac{1}{3^{n_p}} - \frac{1}{3^{n_p+1}}\right). \qquad (6.2.3)$$

Note that due to the definition of μ and the strong separation condition over the IFS $\{f_1, f_2, f_3, f_4\}$, it is clear that $\mu(f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}(E_0)) = \dfrac{1}{4^n}$.

Taking μ of both sides of Equation (6.2.3) and dividing by $(2r_p)^s$, we have

$$\begin{aligned} \frac{\mu(B(x, r_p))}{(2r_p)^s} &\leq \frac{\mu(B(y_{n_p}, 3^{-n_p} - 3^{-(n_p+1)}))}{(2r_p)^s} \\ &= \frac{4^{-(n_p+1)}}{2^s(3^{-n_p} - 3^{-(n_p+1)} - 3^{-(n_p+k)}\sqrt{2})^s} \\ &= \frac{1}{4 \cdot 2^s(1 - 3^{-1} - 3^{-k}\sqrt{2})^s} \end{aligned}$$

Note that $x \in (C \times C) \cap \left(\bigcap_{p \geq 1} F_{p,k}\right)$. Letting $p \to \infty$, we have

$$\underline{D}^s(\mu, x) \leq \frac{1}{4 \cdot 2^s(1 - 3^{-1} - 3^{-k}\sqrt{2})^s}$$

for $k \geq 1$. Letting $k \to \infty$,

$$\underline{D}^s(\mu, x) \leq \frac{1}{4 \cdot \frac{4^s}{3^s}} = \frac{1}{4^s}$$

for $x \in (C \times C) \cap \left(\bigcap_{k \geq 1}^{\infty} \left(\bigcap_{p \geq 1}^{\infty} F_{p,k}\right)\right)$.

By Lemma 6.2.10, $\mu\left(\bigcap_{k \geq 1}^{\infty} \left(\bigcap_{p \geq 1}^{\infty} F_{p,k}\right)\right) = 1$, which means that the μ-measure of any other set that intersects $C \times C$ must be zero. Therefore, $\underline{D}^s(\mu, x) \leq \frac{1}{4^s}$ holds for μ-almost all $x \in C \times C$.

"$\geq$" Given $r > 0$, it suffices to show that

$$\frac{\mu(B(A,r))}{(2r)^s} \geq \frac{1}{4^s} \text{ for } \mu\text{-almost all } A \in C \times C.$$

But since $C \times C = \bigcap_{i=1}^4 (f_i(E_0) \cap (C \times C))$, and $f_1(E_0) \cap (C \times C)$, $f_2(E_0) \cap (C \times C)$, $f_3(E_0) \cap (C \times C)$ and $f_4(E_0) \cap (C \times C)$ are all geometrically similar to one another, it is enough to prove that

$$\frac{\mu(B(A,r))}{(2r)^s} \geq \frac{1}{4^s} \text{ for } \mu\text{-almost all } A \in f_1(E_0) \cap (C \times C). \tag{6.2.4}$$

As well as that, since $C \times C$ is self-similar, proving Equation (6.2.4) for when $\frac{\sqrt{2}}{3} < r \leq \sqrt{2}$ is equivalent to proving that

$$\frac{\mu(B(A,r))}{(2r)^s} \geq \frac{1}{4^s} \text{ for } \mu\text{-almost all } A \in f_i^k(E_0) \cap (C \times C),$$

when $\frac{\sqrt{2}}{3^{k+1}} < r \leq \frac{\sqrt{2}}{3^k}$ for all $k > 0$ and $i = 1, 2, 3, 4$. Letting $k \to \infty$, this would account for all possible $A \in f_1(E_0) \cap (C \times C)$, thus giving us Equation (6.2.4) for all $r > 0$.

So we simply need to show Equation (6.2.4) for $\frac{\sqrt{2}}{3} < r \leq \sqrt{2}$.

Of course we know from Lemma 6.2.11 that

$$\lim_{r \to 0} \frac{\mu(B_r(x))}{(2r)^s} = \frac{1}{4^s}$$

for any vertex x of a basic square at any given level of the construction of $C \times C$. Therefore, if we could show that

$$\mu(B(A,r)) \geq \mu(B(O,r)) \text{ for } \mu\text{-almost all } A \in f_1(E_0) \cap (C \times C)$$

when $\frac{\sqrt{2}}{3} < r \leq \sqrt{2}$ and where O refers to the origin $(0,0)$, we would have our result.

We draw a diagonal line in E_0 between $(0,0)$ and $(1,1)$. Because $f_1(E_0) \cap (C \times C)$ is symmetric with respect to this diagonal, it is enough to consider $A \in S_1$, where S_1 is the triangle formed in $f_1(E_0)$ between the diagonal and the x-axis. Given $\frac{\sqrt{2}}{3} < r \leq \sqrt{2}$ we have two distinct cases:

Case 1: $B(O,r) \cap (f_2(E_0) \cup f_3(E_0))$

It is easily seen that

$$B(A,r) \cap (f_2(E_0) \cup f_3(E_0)) \supseteq B(O,r) \cap (f_2(E_0) \cup f_3(E_0)),$$

since the origin O is the furthest point in S_1 from all points in $f_2(E_0) \cup f_3(E_0)$. Therefore,

$$\mu(B(A,r) \cap (f_2(E_0) \cup f_3(E_0))) \geq \mu(B(O,r) \cap (f_2(E_0) \cup f_3(E_0))).$$

Case 2: $B(O,r) \cap f_4(E_0)$

To prove that $\mu(B(A,r) \cap f_4(E_0)) \geq \mu(B(O,r) \cap f_4(E_0))$ is more difficult and requires a good deal of geometrical manipulation. Jia *et al* do this by analysing progressively smaller triangles that sit inside S_1. They show that there is a certain subset of these triangles which does not satisfy the equation, but this subset has zero μ-measure, thus the equation holds for μ-almost all $A \in S_1$.

Thus, given $r > 0$ we have

$$\frac{\mu(B(A,r)}{(2r)^s} \geq \frac{\mu(B(O,r))}{(2r)^s} \quad \text{for } \mu\text{-almost all } A \in C \times C$$

and letting $r \to 0$,

$$\varliminf_{r \to 0} \frac{\mu(B(A,r)}{(2r)^s} \geq \varliminf_{r \to 0} \frac{\mu(B(O,r))}{(2r)^s} = \frac{1}{4^s} \quad \text{for } \mu\text{-almost all } A \in C \times C$$

by Lemma 6.2.11.

□

We may now prove Theorem 6.2.5:

Proof. The result follows easily from Lemmas 6.2.9 and 6.2.12. □

6.2.3 Remarks

As we have seen, it is possible to successfully use lower spherical density and its respective properties to find the packing measure of a set. The particular method shown above extends to a more general class of fractal sets in $\mathbb{R}^2$, as claimed by Jia *et al* in [JZZL03]. Letting $0 < \lambda \leq \frac{1}{3}$ and supposing that $f_1(x) = \lambda x$, $f_2(x) = 1 - \lambda + \lambda x$ where $x \in [0,1]$ and that C_λ is the invariant set associated with the IFS $\{f_1, f_2\}$, then the result

$$\mathcal{P}^{s(\lambda)}(C_\lambda \times C_\lambda) = 4 \cdot 2^{s(\lambda)}(1-\lambda)^{s(\lambda)}$$

can be achieved, where $s(\lambda) = \log_{\frac{1}{\lambda}} 4$.

Interestingly, Jia *et al* also note that their method cannot be used to calculate the packing measure of self-affine sets such as $C_{\frac{1}{4}} \times C_{\frac{1}{3}}$.

In general, it seems to be easier to calculate results for packing measure using local properties than to do so for Hausdorff measure. This is largely due to results like Lemma 6.2.8 which can directly relate local spherical density to the measure being used. The more useful local density results for Hausdorff measure rely on convex density as opposed to spherical density and obviously it is easier to work with balls than with convex sets when attempting calculations involving coverings or packings.

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